V.

A Catalogue of 130 Polar Stars for the Epoch of 1875.0, resulting from all the available Observations made between 1860 and 1885, and reduced to the System of the Catalogue of Publication XIV. of the Astronomische Gesellschaft.

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FIRST PAPER.

Note by William A. Rogers. — My connection with this work is limited to the methods of discussion adopted, and to an examination of the numerical results obtained. Beyond this, all the work in the preparation of this paper has been done by my assistant, Miss Winlock, and she is entitled to all the credit therefor.

It is the purpose of the present paper to discuss the modern observations of such polar stars north of $+70^{\circ}$ declination as are found in the Harvard College Catalogue of 1213 Stars. Of the 130 stars in this list, 68 are found in the Fundamental Catalogue of Dr. Auwers.

There will be no attempt to determine the proper motions of these stars, but the places determined for the epoch 1875.0 will serve a useful purpose in future discussions of this element. The provisional values of μ and μ' employed in the reductions are those given in the Harvard College Catalogue.

All the observations employed in this discussion will be reduced to the System of Publication XIV. of the Astronomische Gesellschaft, either directly through the medium of the fundamental stars common to the observed and the fundamental systems, or indirectly through the medium of the Harvard College Catalogue.

For many reasons, it was found advisable to construct a yearly ephemeris of each of the stars in the proposed list, extending from 1860 to 1885. For the fundamental stars of the list, the places for 1880 and for subsequent years were taken directly either from the catalogues of the Astronomische Gesellschaft or from the

Berliner Jahrbuch. For the years 1871 to 1879 inclusive, they were obtained by applying to the yearly ephemerides of the Gesellschaft the corresponding corrections by which the provisional system is reduced to the system of Publication XIV.

For the places of the fundamental stars between 1860 and 1870, and for the places of all non-fundamental stars for the entire period between 1860 and 1885, the reduction-elements given in the Harvard College Catalogue were employed.

For stars below 85° north declination the development of α and δ in terms of the first, second, and third powers of the time will be sufficiently accurate for the limit of fifteen years. For the reduction of stars near the pole, the problem becomes more difficult. Since the method of development by differential coefficients in terms of the ascending powers of the time has necessary limitations in its application, it has been thought advisable to give an illustration of the various methods by which the co-ordinates for any time t_0 are reduced to those for any time t'. The star Groombridge 1119 is selected for this purpose. The reductions for precession and for proper motion will be considered independently.

Reduction of the Right Ascension and the Declination for the Equator and Equinox of any Time t_0 to the Values for any Time t' by the Trigonometrical Method of Bohnenberger.

From Bessel's Tabulæ Regiomontanæ, pp. vii, viii, we have

$$\left.\begin{aligned} \cos\delta'\sin(\alpha'+\lambda'-z') &= \cos\delta\sin(\alpha+\lambda+z) \\ \cos\delta'\cos(\alpha'+\lambda'-z') &= \cos\delta\cos(\alpha+\lambda+z)\cos\theta-\sin\delta\sin\theta \\ \sin\delta' &= \cos\delta\cos(\alpha+\lambda+z)\sin\theta+\sin\delta\cos\theta \end{aligned}\right\} \tag{1}$$

Writing for brevity

$$A=\alpha+\lambda+z, \qquad A'=\alpha'+\lambda'-z',$$

we have

$$\left.\begin{aligned} \cos\delta'\sin A' &= \cos\delta\sin A \\ \cos\delta'\cos A' &= \cos\delta\cos A\cos\theta-\sin\delta\sin\theta \\ \sin\delta' &= \cos\delta\cos A\sin\theta+\sin\delta\cos\theta \end{aligned}\right\} \tag{2}$$

From the first two equations of (2) we obtain

$$\left.\begin{aligned} \cos\delta'\sin(A'-A) &= \cos\delta\sin A\sin\theta\,[\tan\delta+\tan\tfrac{1}{2}\theta\cos A] \\ \cos\delta'\cos(A'-A) &= \cos\delta-\cos\delta\cos A\sin\theta\,[\tan\delta+\tan\tfrac{1}{2}\theta\cos A] \end{aligned}\right\} \tag{3}$$

If we put

$$p=\sin\theta\,[\tan\delta+\tan\tfrac{1}{2}\theta\cos A],$$

we have

$$\tan(A'-A)=\frac{p\sin A}{1-p\cos A}, \tag{4}$$

and

$$\tan\tfrac{1}{2}(\delta'-\delta)=\tan\tfrac{1}{2}\theta\,\frac{\cos\frac{1}{2}(A'+A)}{\cos\frac{1}{2}(A'-A)}. \tag{5}$$

Encke has suggested a form of equation (4) which is well adapted for logarithmic computation. (See Danckwortt, Vierteljahrsschrift der Astronomischen Gesellschaft, XVI., 1881, p. 10.)

Let

$$m \sin M = \sin \delta,$$
$$m \cos M = \cos \delta \cos A,$$

whence

$$\tan M = \frac{\tan \delta}{\cos A}.$$

Substituting in (2) we have

$$\begin{aligned} \cos \delta' \sin A' &= \cos \delta \sin A, \\ \cos \delta' \cos A' &= m \cos (M + \theta), \\ \sin \delta' &= m \sin (M + \theta), \end{aligned}$$

whence

$$\left.\begin{aligned} \tan A' &= \frac{\tan A \cos M}{\cos (M + \theta)} \\ \tan \delta' &= \tan (M + \theta) \cos A' \end{aligned}\right\} \tag{6}$$

Then

$$\alpha' = A' + (z' - \lambda').$$

For the computation of the quantities z, z', and θ, we have the following general equations (see Chauvenet, Vol. I. p. 613):

$$\left.\begin{aligned} \cos \tfrac{1}{2} \theta \sin \tfrac{1}{2} (z' + z) &= \sin \tfrac{1}{2} (\psi' - \psi) \cos \tfrac{1}{2} (\varepsilon_1' + \varepsilon_1) \\ \cos \tfrac{1}{2} \theta \cos \tfrac{1}{2} (z' + z) &= \cos \tfrac{1}{2} (\psi' - \psi) \cos \tfrac{1}{2} (\varepsilon_1' - \varepsilon_1) \\ \sin \tfrac{1}{2} \theta \sin \tfrac{1}{2} (z' - z) &= \cos \tfrac{1}{2} (\psi' - \psi) \sin \tfrac{1}{2} (\varepsilon_1' - \varepsilon_1) \\ \sin' \tfrac{1}{2} \theta \cos \tfrac{1}{2} (z' - z) &= \sin \tfrac{1}{2} (\psi' - \psi) \sin \tfrac{1}{2} (\varepsilon_1' + \varepsilon_1) \end{aligned}\right\} \tag{7}$$

whence

$$\left.\begin{aligned} \tan \tfrac{1}{2} (z' + z) &= \tan \tfrac{1}{2} (\psi' - \psi) \cos \tfrac{1}{2} (\varepsilon_1' + \varepsilon_1) \\ \tfrac{1}{2} (z' - z) &= \frac{\tfrac{1}{2} (\varepsilon_1' - \varepsilon_1)}{\tan \tfrac{1}{2} (\psi' - \psi) \sin \tfrac{1}{2} (\varepsilon_1' + \varepsilon_1)} \\ \sin \tfrac{1}{2} \theta &= \sin \tfrac{1}{2} (\psi' - \psi) \sin \tfrac{1}{2} (\varepsilon_1' + \varepsilon_1) \end{aligned}\right\} \tag{8}$$

In which the constants of Struve and Peters for the epoch 1800 are to be employed. They are as follows:

$$\varepsilon_0 = 23^\circ\ 27'\ 54''.22$$

$$\varepsilon_1 = \varepsilon_0 + 0''.00000735\, t^2$$

$$\varepsilon = \varepsilon_0 - 0''.4738\, t - 0''.0000014\, t^2$$

$$\lambda = 0''.15119\, t - 0''.00024186\, t^2$$

$$\psi = 50''.3798\, t - 0''.0001084\, t^2$$

COMPUTATION OF $z' - \lambda'$, $z + \lambda$, AND θ, FOR 1867 AND FOR 1883.

	1867.	1875.	1883.
t	+67	+75	+83
$\log t$	1.8260748	1.8750613	1.9190781
$\log t^2$	3.6521496	3.7501226	3.8381562
	° ′ ″	° ′ ″	° ′ ″
ε_0	23 27 54.220000	23 27 54.220000	23 27 54.220000
$+.00000735\,t^2$	+.032994	+.041344	+.050634
ε_1'	23 27 54.252994	ε_1 23 27 54.261344	ε_1' 23 27 54.270634
$50.3798\,t$	0 56 15.4466	1 2 58.4850	1 9 41.5234
$-.0001084\,t^2$	−.4866	−.6097	− .7468
ψ'	0 56 14.9600	ψ 1 2 57.8753	ψ' 1 9 40.7766
$+.15119\,t$	+10.1297	+11.3392	+12.5487
$-.00024186\,t^2$	− 1.0857	− 1.3605	− 1.6662
λ'	+ 9.0440	λ + 9.9787	λ' +10.8825

	For 1867.	For 1883.
	° ′ ″	° ′ ″
$\frac{1}{2}(\psi' - \psi)$	−0 3 21.4576	+0 3 21.4506
$\frac{1}{2}(\varepsilon_1' + \varepsilon)$	23 27 54.2572	23 27 54.2660
$\frac{1}{2}(\varepsilon_1' - \varepsilon_1)$	−0.004175	+0.004645
(1) $\log \sin \frac{1}{2}(\psi' - \psi)$	6.9897585n	6.9897434
(2) $\log \tan \frac{1}{2}(\psi' - \psi)$	6.9897587n	6.9897436
(3) $\log \sin \frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	9.6000903	9.600903
(4) $\log \cos \frac{1}{2}(\varepsilon_1' + \varepsilon_1)$	9.9625128	9.9625128
(2) + (4) = $\log \tan \frac{1}{2}(z' + z)$	6.9522715n	6.9522564
(5) = $\log \frac{1}{2}(\varepsilon_1' - \varepsilon_1)$	7.6206565n	7.6669857
(6) = (2) + (3)	6.5898490n	6.5898339
(5) − (6) = $\log \frac{1}{2}(z' - z)$	1.0308075	1.0771518
	° ′ ″	° ′ ″
$\frac{1}{2}(z' + z)$	−0 3 4.7972	+0 3 4.7908
$\frac{1}{2}(z' - z)$	+10.7351	+11.9441
z'	−0 2 54.0621	+0 3 16.7349
λ'	− 9.0440	−10.8826
$z' - \lambda'$	−0 3 3.106	+0 3 5.852
z	−0 3 15.5323	+0 2 52.8467
λ	+ 9.9787	+ 9.9787
$z + \lambda$	−0 3 5.554	+0 3 2.825
(1) + (3) = $\log \sin \frac{1}{2}\theta$	6.5898488n	6.5898337
	° ′ ″	° ′ ″
$\frac{1}{2}\theta$	−0 1 20.2183	+0 1 20.2156
θ	−0 2 40.437	+0 2 40.431

TABULAR VALUES OF THE CONSTANTS ϵ_1, ψ, AND λ, FOR THE EPOCHS $1800 \pm t$.

Epoch.	ϵ_1	Δ_1	Δ_1	ψ	Δ_1	Δ_2	λ	Δ_2	Δ_2
	° ′ ″	″	″	° ′ ″	′ ″	″	″	″	″
1755	+23 27 54.234884			−0 37 47.3105			− 7.2931		
		−.004822			+6 43.1095			+1.3682	
1763	+23 27 54.230062		+.000941	−0 31 4.2010		−.0139	− 5.9249		−.0310
		−.003881			+6 43.0956			+1.3372	
1771	+23 27 54.226181		+.000941	−0 24 21.1054		−.0138	− 4.5877		−.0309
		−.002940			+6 43.0818			+1.3063	
1779	+23 27 54.223241		+.000941	−0 17 38.0236		−.0139	− 3.2814		−.0311
		−.001999			+6 43.0679			+1.2752	
1787	+23 27 54.221242		+.000941	−0 10 54.9557		−.0139	− 2.0062		−.0309
		−.001058			+6 43.0540			+1.2443	
1795	+23 27 54.220184		+.000940	−0 4 11.9017		−.0139	− 0.7619		−.0309
		−.000118			+6 43.0401			+1.2134	
1803	+23 27 54.220066		+.000941	+0 2 31.1384		−.0138	+ 0.4515		−.0310
		+.000823			+6 43.0263			+1.1824	
1811	+23 27 54.220889		+.000941	+0 9 14.1647		−.0139	+ 1.6339		−.0309
		+.001764			+6 43.0124			+1.1515	
1819	+23 27 54.222653		+.000941	+0 15 57.1771		−.0139	+ 2.7854		−.0310
		+.002705			+6 42.9985			+1.1205	
1827	+23 27 54 225358		+.000941	+0 22 40.1756		−.0139	+ 3.9059		−.0310
		+.003646			+6 42.9846			+1.0895	
1835	+23 27 54.229004		+.000940	+0 29 23.1602		−.0139	+ 4.9954		−.0309
		+.004586			+6 42.9707			+1.0586	
1843	+23 27 54.233590		+.000941	+0 36 6.1309		−.0138	+ 6.0540		−.0310
		+.005527			+6 42.9569			+1.0276	
1851	+23 27 54.239117		+.000941	+0 42 49.0878		−.0138	+ 7.0816		−.0309
		+.006468			+6 42.9431			+0.9967	
1859	+23 27 54.245585		+.000941	+0 49 32.0309		−.0140	+ 8.0783		−.0310
		+.007409			+6 42.9291			+0.9657	
1867	+23 27 54.252994		+.000941	+0 56 14.9600		−.0138	+ 9.0440		−.0310
		+.008350			+6 42.9153			+0.9347	
1875	+23 27 54.261344		+.000940	+1 2 57.8753		−.0140	+ 9.9787		−.0309
		+.009290			+6 42.9013			+0.9038	
1883	+23 27 54 270634		+.000941	+1 9 40.7766		−.0138	+10.8825		−.0310
		+.010231			+6 42.8875			+0.8728	
1891	+23 27 54.280865		+.000941	+1 16 23.6641		−.0138	+11.7553		−.0309
		+.011172			+6 42.8737			+0.8419	
1899	+23 27 54.292037		+.000941	+1 23 6.5378		−.0140	+12.5972		−.0310
		+.012113			+6 42.8597			+0.8109	
1907	+23 27 54.304150		+.000941	+1 29 49.3975		−.0138	+13.4081		−.0310
		+.013054			+6 42.8459			+0.7799	
1915	+23 27 54.317204		+.000940	+1 36 32 2434		−.0139	+14.1880		−.0309
		+.013994			+6 42.8320			+0.7490	
1923	+23 27 54.331198		+.000941	+1 43 15.0754		−.0139	+14.9370		−.0310
		+.014935			+6 42.8181			+0.7180	
1931	+23 27 54.346133		+.000941	+1 49 57.8935		−.0138	+15.6550		−.0309
		+.015876			+6 42.8043			+0.6871	
1939	+23 27 54.362009		+.000941	+1 56 40.6978		−.0138	+16.3421		−.0310
		+.016817			+6 42.7904			+0.6561	
1947	+23 27 54.378826		+.000941	+2 3 23.4882		−.0139	+16.9982		−.0309
		+.017758			+6 42.7765			+0.6252	
1955	+23 27 54.396584			+2 10 6.2647			+17.6234		

TABULAR VALUES OF THE CONSTANTS $z' - \lambda'$, $z + \lambda$, AND θ, FOR THE EPOCHS $1800 \pm t$.

Epoch.	$z' - \lambda'$	Δ_1	Δ_2	$z + \lambda$	Δ_1	Δ_2	θ	Δ_1	Δ_2
	° ′ ″	′ ″	″	° ′ ″	′ ″	″	° ′ ″	′ ″	″
1755	—0 46 3.1047			—0 46 4.9535			—0 40 7.0594		
		+3 4.1291			+3 4.2880			+2 40.5008	
1763	—0 42 58.9756		+.0241	—0 43 0.6655		—.0072	—0 37 26.5586		—.0042
		+3 4.1532			+3 4.2808			+2 40.4966	
1771	—0 39 54.8224		+.0237	—0 39 56.3847		—.0069	—0 34 46.0620		—.0036
		+3 4.1769			+3 4.2739			+2 40.4930	
1779	—0 36 50.6455		+.0241	—0 36 52.1108		—.0071	—0 32 5.5690		—.0044
		+3 4.2010			+3 4.2668			+2 40.4886	
1787	—0 33 46.4445		+.0238	—0 33 47.8440		—.0072	—0 29 25.0804		—.0036
		+3 4.2248			+3 4.2596			+2 40.4850	
1795	—0 30 42.2197		+.0244	—0 30 43.5844		—.0071	—0 26 44.5954		—.0038
		+3 4.2492			+3 4.2525			+2 40.4812	
1803	—0 27 37.9705		+.0238	—0 27 39.3319		—.0069	—0 24 4.1142		—.0047
		+3 4.2730			+3 4.2456			+2 40.4765	
1811	—0 24 33.6975		+.0240	—0 24 35.0863		—.0067	—0 21 23.6377		—.0050
		+3 4.2970			+3 4.2389			+2 40.4715	
1819	—0 21 29.4005		+.0241	—0 21 30.8474		—.0068	—0 18 43.1662		—.0045
		+3 4.3211			+3 4.2321			+2 40.4670	
1827	—0 18 25.0794		+.0245	—0 18 26.6153		—.0066	—0 16 2.6992		—.0042
		+3 4.3456			+3 4.2255			+2 40.4628	
1835	—0 15 20.7338		+.0247	—0 15 22.3898		—.0065	—0 13 22.2364		—.0052
		+3 4.3703			+3 4.2190			+2 40.4576	
1843	—0 12 16.3635		+.0246	—0 12 18.1708		—.0064	—0 10 41.7788		—.0052
		+3 4.3949			+3 4.2126			+2 40.4524	
1851	—0 9 11.9686		+.0243	—0 9 13.9582		—.0070	—0 8 1.3264		—.0050
		+3 4.4192			+3 4.2056			+2 40.4474	
1859	—0 6 7.5494		+.0241	—0 6 9.7526		—.0066	—0 5 20.8790		—.0058
		+3 4.4433			+3 4.1990			+2 40.4416	
1867	—0 3 3.1061			—0 3 5.5536			—0 2 40.4374		
1875	. . . . .	. . .	. . .	. . . . . .	. . .	. . .	. . . . .	. . . . .	. . .
1883	+0 3 5.8523			+0 3 2.8254			+0 2 40.4312		
		+3 4.5154			+3 4.1845			+2 40.4262	
1891	+0 6 10.3677		+.0277	+0 6 7.0099		—.0137	+0 5 20.8574		—.0067
		+3 4.5431			+3 4.1708			+2 40.4195	
1899	+0 9 14.9108		+.0244	+0 9 11.1807		—.0048	+0 8 1.2769		—.0058
		+3 4.5675			+3 4.1660			+2 40.4137	
1907	+0 12 19.4783		+.0247	+0 12 15.3467		—.0050	+0 10 41.6906		—.0059
		+3 4.5922			+3 4.1610			+2 40.4078	
1915	+0 15 24.0705		+.0242	+0 15 19.5077		—.0053	+0 13 22.0984		—.0059
		+3 4.6164			+3 4.1557			+2 40.4019	
1923	+0 18 28.6869		+.0248	+0 18 23.6634		—.0061	+0 16 2.5003		—.0060
		+3 4.6412			+3 4.1496			+2 40.3959	
1931	+0 21 33.3281		+.0249	+0 21 27.8130		—.0061	+0 18 42.8962		—.0067
		+3 4.6661			+3 4.1435			+2 40.3892	
1939	+0 24 37.9942		+.0252	+0 24 31.9565		—.0063	+0 21 23.2854		—.0066
		+3 4.6913			+3 4.1372			+2 40.3826	
1947	+0 27 42.6855		+.0251	+0 27 36.0937		—.0065	+0 24 3.6680		—.0068
		+3 4.7164			+3 4.1307			+2 40.3758	
1955	+0 30 47.4019			+0 30 40.2244			+0 26 44.0438		

The following example is given in illustration of the application of equations (6), the problem being to reduce from the mean equator and equinox of 1875.0 to the mean equator and equinox of 1867.0 and of 1883.0. Given the co-ordinates of Groombridge 1119 for 1875.0, to find the values for 1867.0 and 1883.0.

α for 1875.0 = 7^h 29^m 5^s.631 δ for 1875.0 = +88° 59′ 37″.69

	For 1867.0.	For 1883.0.
	° ′ ″	° ′ ″
α 1875.0	112 16 24.465	112 16 24.465
$z + \lambda$	−0 3 5.554	+0 3 2.825
A	112 13 18.911	112 19 27.290
log tan δ	1.7553949	1.7553949
log cos A	9.5777158n	9.5796095n
log tan M	2.1776791n	2.1757854n
	° ′ ″	° ′ ″
M	+90 22 50.060	+90 22 56.048
θ	− 0 2 40.437	+ 0 2 40.431
$M + \theta$	+90 20 9.623	+90 25 36.479
log tan A	0.3887667n	0.3865553n
log cos M	7.8223112n	7.8242051n
log tan A cos M	8.2110779	8.2107604
log cos $(M + \theta)$	7.7682224n	7.8720975n
log tan A'	0.4428555n	0.3386629n
	° ′ ″	° ′ ″
A'	+109 50 3.500	+114 37 52.839
$z' - \lambda'$	− 0 3 3.106	+ 0 3 5.852
	° ′ ″	° ′ ″
α' in arc	+109 47 0.394	+114 40 58.691
	h. m. s.	h. m. s.
α' in time	7 19 8.026	7 38 43.913
log tan $(M + \theta)$	2.2317701n	2.1278903n
log cos A'	9.5305854n	9.6199049n
log tan δ'	1.7623555	1.7477952
	° ′ ″	° ′ ″
δ'	+89 0 35.27	+88 58 33.76

In like manner, the following values of α and δ were computed for intervals of eight years.

Date.	α	δ	Date.	α	δ
	h. m. s.	° ′ ″		h. m. s.	° ′ ″
1755	4 45 55.203	+89 1 1.38	1859	7 8 52.315	+89 1 26.19
1763	4 56 19.577	+89 1 48.90	1867	7 19 8.026	+88 0 35.27
1771	5 6 59.542	+89 2 29.33	1875	7 29 5.631	+88 59 37.69
1779	5 17 52.998	+89 3 2.41	1883	7 38 43.913	+88 58 33.76
1787	5 28 57.519	+89 3 27.93	1891	7 48 1.994	+88 57 23.81
1795	5 40 10.397	+89 3 45.70	1899	7 56 59.309	+88 56 8.17
1803	5 51 28.695	+89 3 55.62	1907	8 5 35.579	+88 54 47.17
1811	6 2 49.333	+89 3 57.66	1915	8 13 50.774	+88 53 21.14
1819	6 14 9.166	+89 3 51.70	1923	8 21 45.073	+88 51 50.40
1827	6 25 25.071	+89 3 37.84	1931	8 29 18.828	+88 50 15.24
1835	6 36 34.034	+89 3 16.18	1939	8 36 32.527	+88 48 35.97
1843	6 47 33.225	+89 2 46.91	1947	8 43 26.768	+88 46 52.87
1851	6 58 20.073	+89 2 10.16	1955	8 50 2.231	+88 45 6.21

The places for single years will be easily obtained by successive interpolations to the middle, by means of the formula (see Chauvenet, Vol. I. p. 88),

$$F^{\frac{1}{2}} = \tfrac{1}{2}\,(F + F') - \tfrac{1}{8}\left[b_0 - \tfrac{3}{16}\left[d_0 - \tfrac{5}{24}\,(f_0 - \&c.)\right]\right] \tag{9}$$

Bessel has given a method (see Wolfers, Tab. Reg., pp. lii, liii) by which the true values of α' and δ' may be computed from an approximate value of α'.

Let

$$\alpha_1' = \text{an approximate value of } \alpha',$$

$$A_1' = \alpha_1' - (z' - \lambda').$$

From Napier's first and second Analogies,

$$\tan \tfrac{1}{2}\,(\delta' - \delta) = \frac{\cos \frac{1}{2}\,(A_1' + A)}{\cos \frac{1}{2}\,(A_1' - A)} \tan \tfrac{1}{2}\,\theta \tag{10}$$

$$\operatorname{cotan} \tfrac{1}{2}\,(\delta' - \delta) = \frac{\sin \frac{1}{2}\,(A_1' + A)}{\sin \frac{1}{2}\,(A_1' - A)} \tan \tfrac{1}{2}\,\theta \tag{11}$$

Let the variations in the logarithms of the given functions for a change of 1″ have the following designations:

$\Delta \log \cos \frac{1}{2}(A_1' + A) = c$ $\qquad$ $\Delta \log \sin \frac{1}{2}(A_1' - A) = s'$

$\Delta \log \cos \frac{1}{2}(A_1' - A) = c'$ $\qquad$ $\Delta \log \tan \frac{1}{2}(\delta' - \delta) = t$

$\Delta \log \sin \frac{1}{2}(A_1' + A) = s$ $\qquad$ $\Delta \log \tan \frac{1}{2}(\delta' + \delta) = t'$

d = the value of $\frac{1}{2}(\delta' - \delta)$ obtained from equation (10)

d' = " " $\frac{1}{2}(\delta' + \delta)$ " " " (11)

x = the required correction to α_1' in order to obtain α'

$$\beta = \frac{\frac{1}{2}(c - c')}{t} \qquad \gamma = \frac{\frac{1}{2}(\delta - \delta')}{t'}$$

Then

$$x = \frac{\delta - (d' - d)}{\gamma - \beta}$$

$$\alpha' = \alpha_1' + x. \tag{12}$$

$$\delta' = d' + d + (\beta + \gamma)\, x. \tag{13}$$

This method may be conveniently applied when we have a series of values of α and require any following term of the series. From the differences of the given values of α an approximation to the true value may be obtained, from which the correction x can be computed. The exact values of α' and δ' will then be determined from equations (12) and (13) respectively.

Let us assume that the values of α have been found for 1875, 1883, and 1891 as follows:

		Δ_1	Δ_2	
	h. m. s.	m. s.	s.	° ′ ″
1875	7 29 5.631			$\delta = +88\ 59\ 37.69$
		+9 38.282		
1883	7 38 43.913		−20.201	
		+9 18.081		
1891	7 48 1.994			

Assuming the second difference, $-20^s.201$, as a constant, we obtain, for 1899,

$$\alpha_1' = 7^h\ 56^m\ 59^s.874$$

To find the true α' for 1899, we have

	h. m. s.		
α_1'	7 56 59.874	$\log \cos \frac{1}{2}(A_1' + A)$	9.6381071n
	° ′ ″	$\log \cos \frac{1}{2}(A_1' - A)$	9.9992641
α_1' in arc	+119 14 58.110	$\log \frac{\cos \frac{1}{2}(A_1' + A)}{\cos \frac{1}{2}(A_1' - A)}$	9.6388430n
$z' - \lambda'$	+ 0 9 14.911	$\log \tan \frac{1}{2}\theta$	7.0669400
A_1'	+119 5 43.199	$\log \tan \frac{1}{2}(\delta' - \delta)$	6.7057830n
α	+112 16 24.465		
$z + \lambda$	+ 0 9 11.181	$\log \sin \frac{1}{2}(A_1' + A)$	9.9545393
A	+112 25 35.646	$\log \sin \frac{1}{2}(A_1' - A)$	8.7646475
$\frac{1}{2}(A_1' + A)$	+115 45 39.422	$\log \frac{\sin \frac{1}{2}(A_1' + A)}{\sin \frac{1}{2}(A_1' - A)}$	1.1898918
$\frac{1}{2}(A_1' - A)$	+ 3 20 3.776	$\log \tan \frac{1}{2}\theta$	7.0669400
$\frac{1}{2}\theta$	+ 0 4 0.6385	$\log \text{cotan} \frac{1}{2}(\delta' + \delta)$	8.2568318

$c = +43.6$	$s = -\ 10.2$	$t\ -41660$	$d\ -\ 0°\ 1'\ 44.763''$
$c' = -\ 1.2$	$s' = +361.3$	$t'\ -\ 1166.1$	$d'\ +88\ 57\ 54.285$

$$\beta = \tfrac{1}{2}\left(\frac{+43.6 + 1.2}{-41660}\right) = -0.00054 \qquad \gamma = \tfrac{1}{2}\left(\frac{-10.2 - 361.3}{1166.1}\right) = +0.15930$$

$$\gamma - \beta = +0.15984 \qquad \beta + \gamma = +0.15876$$

$$x = \frac{88°\ 59'\ 37.69'' - 88°\ 59'\ 39.048''}{+0.15984} = -8.496''$$

	h. m. s.		
$\alpha_1' =$	7 56 59.874	$\log(\beta + \gamma)$	9.2007411
$x =$	−0.566	$\log x$	0.9292145n
$\alpha' =$	7 56 59.308	$\log(\beta + \gamma)\,x$	0.1299556n
			° ′ ″
		$(\beta + \gamma)\,x$	−1.349
		$d' + d$	88 56 9.522
		δ'	88 56 8.17

Development of the Functions α and δ by Means of Differential Coefficients, expressed in Terms of the Ascending Powers of the Time.

Given α_0 and δ_0 for any time t_0, to obtain α and δ for any time t', we have, by Taylor's Theorem,

$$\alpha = \alpha_0 + \frac{d\alpha}{dt}(t' - t_0) + \frac{1}{2}\frac{d^2\alpha}{dt^2}(t' - t_0)^2 + \frac{1}{2.3}\frac{d^3\alpha}{dt^3}(t' - t_0)^3 + \frac{1}{2.3.4}\frac{d^4\alpha}{dt^4}(t' - t_0)^4 +, \&c. \qquad (14)$$

$$\delta = \delta_0 + \frac{d\delta}{dt}(t' - t_0) + \frac{1}{2}\frac{d^2\delta}{dt^2}(t' - t_0)^2 + \frac{1}{2.3}\frac{d^3\delta}{dt^3}(t' - t_0)^3 + \frac{1}{2.3.4}\frac{d^4\delta}{dt^4}(t' - t_0)^4 +, \&c. \qquad (15)$$

We must now find expressions convenient for the numerical computation of the differential coefficients $\frac{d\alpha}{dt}, \frac{d^2\alpha}{dt^2}, \frac{d^3\alpha}{dt^3}, \frac{d^4\alpha}{dt^4}$, &c.

The form of development given by Bessel is at once the earliest and the most complete published, since it is carried to the fourth power of the time: it is also quite as well adapted for logarithmic computation as the more modern forms.

Introducing into the denominators that power of the radius which will render all the terms homogeneous, we have (see Tab. Reg., pp. x, xi, and Fundamenta Astronomiæ, p. 301), after multiplying by the coefficients in the development by Taylor's Theorem,

$$\frac{d\alpha}{dt} = m + n \tan\delta \sin\alpha. \tag{16}$$

$$\frac{d^2\alpha}{dt^2} = m' + \frac{n^2}{R}\tan^2\delta \sin 2\alpha + \frac{n\,m}{R}\tan\delta\cos\alpha + \frac{n^2}{2R}\sin 2\alpha + n'\tan\delta\sin\alpha. \tag{17}$$

$$\begin{aligned}\frac{d^3\alpha}{dt^3} = \frac{n^2 m}{2R^2} + \frac{2n^3}{R^2}\tan^3\delta\sin 3\alpha + \frac{3n^2 m}{R^2}\tan^2\delta\cos 2\alpha + \frac{3n^3}{2R^2}\tan\delta\sin 3\alpha + \left(\frac{n^3}{2R^2} - \frac{n\,m^2}{R^2}\right)\tan\delta\sin\alpha \\ + \frac{3n^2 m}{2R^2}\cos 2\alpha + \frac{m'n + 2n'm}{R}\tan\delta\cos\alpha + \frac{3n'n}{R}\tan^2\delta\sin 2\alpha + \frac{3n'n}{2R}\sin 2\alpha.\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{d^4\alpha}{dt^4} = \frac{6n^4}{R^3}\tan^4\delta\sin 4\alpha + \frac{{}^*12n^3 m}{R^3}\tan^3\delta\cos 3\alpha + \frac{6n^4}{R^3}\tan^2\delta\sin 4\alpha + \left(\frac{2n^4}{R^3} - \frac{7n^2m^2}{R^3}\right)\tan^2\delta\sin 2\alpha \\ + \frac{9n^3 m}{R^3}\tan\delta\cos 3\alpha + \left(\frac{2n^3 m}{R^3} - \frac{n\,m^3}{R^3}\right)\tan\delta\cos\alpha + \frac{3n^4}{4R^3}\sin 4\alpha + \left(\frac{n^4}{R^3} - \frac{7n^2m^2}{2R^3}\right)\sin 2\alpha,\end{aligned} \tag{19}$$

&c., &c., &c.

$$\frac{d\delta}{dt} = n\cos\alpha. \tag{20}$$

$$\frac{d^2\delta}{dt^2} = -\frac{n^2}{R}\tan\delta\sin^2\alpha - \frac{n\,m}{R}\sin\alpha + n'\cos\alpha. \tag{21}$$

$$\begin{aligned}\frac{d^3\delta}{dt^3} = -\frac{3n^3}{R^2}\tan^2\delta\sin^2\alpha\cos\alpha - \frac{3n^2 m}{R^2}\tan\delta\sin\alpha\cos\alpha - \left[\frac{n(m^2+n^2)}{R^2}\cos\alpha - \frac{n^3}{R^2}\cos^3\alpha\right] \\ - \left(\frac{m'n + 2n'm}{R}\right)\sin\alpha - \frac{2n'n}{R}\tan\delta\sin^2\alpha.\end{aligned} \tag{22}$$

* Engelmann (see Abhandlungen von Bessel, Vol. I. p. 277) gives $\frac{8n^3m}{R^3}$ for this term.

$$\frac{d^4\delta}{dt^4} = \frac{3n^4}{R^3}\tan^3\delta\,[\sin^4\alpha - 4\sin^2\alpha\cos^2\alpha] + \frac{6n^3m}{R^3}\tan^2\delta\sin\alpha + \frac{18n^3m}{R^3}\tan^2\delta\sin\alpha\cos^2\alpha$$

$$+ \frac{7n^2m^2}{R^3}\tan\delta\sin^2\alpha - \frac{3n^2m^2}{R^3}\tan\delta + \frac{n^4}{R^3}\tan\delta\,(\sin^4\alpha - 8\sin^2\alpha\cos^2\alpha) \tag{23}$$

$$+ \left[\frac{nm(m^2+n^2)}{R^3}\sin\alpha - \frac{6n^3m}{R^3}\sin\alpha\cos^2\alpha\right]$$

&c., &c., &c.

For the computation of the numerical coefficients, we have the following constants for 1875.0. (See Pub. XIV., p. 51.)

For the reduction in Right Ascension,

$m = \overset{s.}{3.072245}$	$n = \overset{s.}{1.336949}$	$m' = +\overset{s.}{0.0000189933}$	
$\log m = 0.4874558$	$\log n = 0.1261147$	$\log m' = 5.2786011$	$\log R = \overset{s.}{4.1383338}$
$\log m^2 = 0.9749116$	$\log n^2 = 0.2522294$		$\log R^2 = 8.2766676$
$\log m^3 = 1.4623674$	$\log n^3 = 0.3783441$	$n' = -0.00000575333$	$\log R^3 = 12.4150014$
	$\log n^4 = 0.5044588$	$\log n' = 4.7599145n$	

For the reduction in Declination,

$m = \overset{''}{46.08367}$	$n = \overset{''}{20.05423}$	$m' = +\overset{''}{0.0002849}$	
$\log m = 1.6635471$	$\log n = 1.3022060$	$\log m' = 6.4546924$	$\log R = \overset{''}{5.3144251}$
$\log m^2 = 3.3270942$	$\log n^2 = 2.6044120$		$\log R^2 = 10.6288502$
	$\log n^3 = 3.9066180$	$n' = -0.0000863$	$\log R^3 = 15.9432753$
	$\log n^4 = 5.2088240$	$\log n' = 5.9360108n$	

Substituting in the preceding equations, we have:

$$\frac{d\alpha}{dt} = +\overset{s.}{3.072245} + [0.1261147]\tan\delta\sin\alpha. \tag{24}$$

$$\frac{d^2\alpha}{dt^2} = +\overset{s.}{0.00001899} + [6.1138956]\tan^2\delta\sin 2\alpha + [6.4752367]\tan\delta\cos\alpha + [5.8128656]\sin 2\alpha + [4.7599145n]\tan\delta\sin\alpha. \tag{25}$$

$$\frac{d^3\alpha}{dt^3} = +\overset{s.}{0.0000000145} + [2.4027065]\tan^3\delta\sin 3\alpha + [2.9401389]\tan^2\delta\cos 2\alpha + [2.2777678]\tan\delta\sin 3\alpha + [2.7811577n]\tan\delta\sin\alpha + [2.6391089]\cos 2\alpha + [0.8598285n]\tan\delta\cos\alpha + [1.2248167n]\tan^2\delta\sin 2\alpha + [0.9237867]\sin 2\alpha. \tag{26}$$

$$\frac{d^4 \alpha}{d t^4} = {}^{*} + [\bar{8}.8676086] \tan^4 \delta \sin 4 \alpha + [\bar{9}.5299797] \tan^3 \delta \cos 3 \alpha + [\bar{8}.8676086] \tan^2 \delta \sin 4 \alpha$$
$$+ [\bar{9}.6330799n] \tan^2 \delta \sin 2 \alpha + [\bar{9}.4050410] \tan \delta \cos 3 \alpha + [\bar{8}.9667501n] \tan \delta \cos \alpha \qquad (27)$$
$$+ [\bar{7}.9645186] \sin 4 \alpha + [\bar{9}.3320498n] \sin 2 \alpha.$$

$$\frac{d \delta}{d t} = + [1.3022060] \cos \alpha. \qquad (28)$$

$$\frac{d^2 \delta}{d t^2} = + [7.2899869n] \tan \delta \sin^2 \alpha + [7.6513280n] \sin \alpha + [5.9360108n] \cos \alpha. \qquad (29)$$

$$\frac{d^3 \delta}{d t^3} = + [3.7548891n] \tan^2 \delta \sin^2 \alpha \cos \alpha + [4.1162302n] \tan \delta \sin \alpha \cos \alpha + [4.0757367n] \cos \alpha$$
$$+ [3.2777678] \cos^3 \alpha + [2.0359378] \sin \alpha + [2.4009130] \sin^2 \alpha \tan \delta. \qquad (30)$$

$$\frac{d^4 \delta}{d t^4} = + [\bar{9}.7426700] \tan^3 \delta \sin^4 \alpha + [0.3447299n] \tan^3 \delta \sin^2 \alpha \cos^2 \alpha + [0.4050411] \tan^2 \delta \sin \alpha$$
$$+ [0.8821623] \tan^2 \delta \sin \alpha \cos^2 \alpha + [0.8333289] \tan \delta \sin^2 \alpha + [0.4653522n] \tan \delta \qquad (31)$$
$$+ [\bar{9}.2655487] \tan \delta \sin^4 \alpha + [0.1686387n] \tan \delta \sin^2 \alpha \cos^2 \alpha + [0.4248824] \sin \alpha$$
$$+ [0.4050411n] \sin \alpha \cos^2 \alpha.$$

Hill has given a slightly different form of development for $\frac{d^3 \alpha}{d t^3}$ and $\frac{d^3 \delta}{d t^3}$. (See Star Tables of the American Ephemeris, p. xvii.)

Introducing the required power of the radius, we have:

$$\frac{d^2 \alpha}{d t^2} = m' + \frac{n^2}{2 R} \sin 2 \alpha + n' \sin \alpha \tan \delta + \frac{m n}{R} \cos \alpha \tan \delta + \frac{n^2}{R} \sin 2 \alpha \tan^2 \delta. \qquad (32)$$

$$\frac{d^3 \alpha}{d t^3} = \frac{m n^2}{2 R^2} + \frac{3 m n^2}{2 R^2} \cos 2 \alpha + \frac{3 n n'}{2 R} \sin 2 \alpha + \frac{2 n^3 - m^2 n}{R^2} \sin \alpha \tan \delta + \frac{3 n^3}{R^2} \sin \alpha \cos 2 \alpha \tan \delta$$
$$+ \frac{2 m n' + n m'}{R} \cos \alpha \tan \delta + \frac{3 m n^2}{R^2} \cos 2 \alpha \tan^2 \delta + \frac{3 n n'}{R} \sin 2 \alpha \tan^2 \delta \qquad (33)$$
$$+ \frac{2 n^3}{R^2} \sin \alpha \tan^3 \delta + \frac{4 n^3}{R^2} \cos 2 \alpha \tan^3 \delta.$$

$$\frac{d^2 \delta}{d t^2} = - \frac{m n}{R} \sin \alpha + n' \cos \alpha - \frac{n^2}{R} \sin^2 \alpha \tan \delta. \qquad (34)$$

* One dash over the characteristic of the logarithm indicates one completion of the cycle, and two dashes indicate two completions of a cycle.

$$\frac{d^3\delta}{dt^3} = -\left(\frac{2mn' + nm'}{R}\right)\sin\alpha - \frac{m^2 n}{R^2}\cos\alpha - \frac{n^3}{R^2}\sin^2\alpha\cos\alpha - \frac{3mn^2}{2R^2}\sin 2\alpha\tan\delta$$
$$- \frac{3nn'}{R}\sin^2\alpha\tan\delta - \frac{3n^3}{R^2}\sin^2\alpha\cos\alpha\tan^2\delta. \qquad (35)$$

Substituting the preceding numerical values of the constants for 1875.0, and adopting the convenient form which Hill has given on p. xix for $\frac{d^2\alpha}{dt^2}$ and $\frac{d^2\delta}{dt^2}$, the formulæ become:

$$\frac{d^2\alpha}{dt^2} = +\overset{s.}{0}.00003221 + [4.6338048n]\frac{d\alpha}{dt} + [5.9877809]\frac{d\alpha}{dt}\cos\alpha\tan\delta$$
$$+ [4.8116896n]\frac{d\delta}{dt}\sin\alpha\sec^2\delta. \qquad (36)$$

$$\frac{d^3\alpha}{dt^3} = +\overset{s.}{0}.0000000145 + [2.6391089]\cos 2\alpha + [0.9237867n]\sin 2\alpha + [2.6176279n]\sin\alpha\tan\delta$$
$$+ [2.5787978]\sin\alpha\cos 2\alpha\tan\delta + [0.8598285n]\cos\alpha\tan\delta + [2.9401389]\cos 2\alpha\tan^2\delta \qquad (37)$$
$$+ [1.2248167n]\sin 2\alpha\tan^2\delta + [2.4027065]\sin\alpha\tan^3\delta + [2.7037365]\sin\alpha\cos 2\alpha\tan^3\delta.$$

$$\frac{d^2\delta}{dt^2} = + [4.6338048n]\frac{d\delta}{dt} + [7.1638722n]\frac{d\alpha}{dt}\sin\alpha. \qquad (38)$$

$$\frac{d^3\delta}{dt^3} = + [2.0359378]\sin\alpha + [4.0004500n]\cos\alpha + [3.2777678n]\sin^2\alpha\cos\alpha \qquad (39)$$
$$+ [3.8152002n]\sin 2\alpha\tan\delta + [2.4009130]\sin^2\alpha\tan\delta + [3.7548891n]\sin^2\alpha\cos\alpha\tan^2\delta.$$

Menten has introduced a convenient modification of the expressions for $\frac{d^2\alpha}{dt^2}$ and $\frac{d^2\delta}{dt^2}$. (See Astronomische Beobachtungen auf der Sternwarte zu Bonn, VII. pp. 147–148.)

Let

$$A = m' + \frac{n^2}{2R}\sin 2\alpha. \qquad A' = n'\cos\alpha - \frac{mn}{R}\sin\alpha.$$

$$\log B = \log\left[\frac{mn}{R}\cos\alpha + n'\sin\alpha\right]. \qquad \log B' = \log\left(-\frac{n^2}{R}\sin^2\alpha\right). \qquad (40)$$

$$\log C = \log\frac{n^2}{R}\sin 2\alpha.$$

In which A, B, and C are expressed in seconds of time, and A' and B' in seconds of arc. Or,

$$
\begin{aligned}
A &= +\ \overset{s.}{0.00001899} + [5.8128656] \sin 2\alpha. \\
\log B &= + [4.7599145n] \sin \alpha + [6.4752367] \cos \alpha. \\
\log C &= + [6.1138956] \sin 2\alpha. \\
A' &= + [7.6513280n] \sin \alpha + [5.9360108n] \cos \alpha. \\
\log B &= + [7.2899869n] \sin^2 \alpha.
\end{aligned}
\tag{41}
$$

Then,

$$
\begin{aligned}
\frac{d^2 \alpha}{d t^2} &= A + B \tan \delta + C \tan^2 \delta. \\
\frac{d^2 \delta}{d t^2} &= A' + B' \tan \delta.
\end{aligned}
\tag{42}
$$

In like manner Tiele has simplified the computation of $\frac{d^3 \alpha}{d t^3}$ and $\frac{d^3 \delta}{d t^3}$. (See Astronomische Beobachtungen auf der Sternwarte zu Bonn, VII. p. 149.)

He gives the equations:

$$
\begin{aligned}
\frac{d^3 \alpha}{d t^3} = & [-m n^2 + 3 n n' \sin \alpha \cos \alpha + 3 m n^2 \cos^2 \alpha] \\
& + [(m' n + 2 m n') \cos \alpha - n (m^2 + n^2) \sin \alpha + 6 n^3 \sin \alpha \cos^2 \alpha] \tan \delta \\
& + [-3 m n^2 + 6 n n' \sin \alpha \cos \alpha + 6 m n^2 \cos^2 \alpha] \tan^2 \delta \\
& + [-2 n^3 \sin \alpha + 8 n^3 \sin \alpha \cos^2 \alpha] \tan^3 \delta.
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
\frac{d^3 \delta}{d t^3} = & [-(m' n + 2 m n') \sin \alpha - m^2 n \cos \alpha - n^3 \cos \alpha \sin^2 \alpha] \\
& + [-3 n n' \sin^2 \alpha - 3 m n^2 \sin \alpha \cos \alpha] \tan \delta \\
& - 3 n^3 \sin^2 \alpha \cos \alpha \tan^2 \delta.
\end{aligned}
\tag{44}
$$

Assuming

$$
\begin{aligned}
P &= -\frac{m n^2}{R^2} + \frac{3 n n'}{R} \sin \alpha \cos \alpha + \frac{3 m n^2}{R^2} \cos^2 \alpha. \\
P_1 &= + \left(\frac{m' n + 2 m n'}{R}\right) \cos \alpha - \frac{(m n^2 + n^2)}{R^2} \sin \alpha + \frac{6 n^3}{R^2} \sin \alpha \cos^2 \alpha. \\
P_2 &= -\frac{3 m n^2}{R^2} + \frac{6 n n'}{R} \sin \alpha \cos \alpha + \frac{6 m n^2}{R^2} \cos^2 \alpha. \\
P_3 &= -\frac{2 n^3}{R^2} \sin \alpha + \frac{8 n^3}{R^2} \sin \alpha \cos^2 \alpha. \\
Q &= -\left(\frac{m' n + 2 m n'}{R}\right) \sin \alpha - \frac{m^2 n}{R^2} \cos \alpha - \frac{n^3}{R^2} \cos \alpha \sin^2 \alpha. \\
Q_1 &= -\frac{3 n n'}{R} \sin^2 \alpha - \frac{3 m n^2}{R^2} \sin \alpha \cos \alpha. \\
Q_2 &= -\frac{3 n^3}{R^2} \cos \alpha \sin^2 \alpha.
\end{aligned}
\tag{45}
$$

Then,

$$\frac{d^3 \alpha}{d t^3} = P + P_1 \tan \delta + P_2 \tan^2 \delta + P_3 \tan^3 \delta.$$

$$\frac{d^3 \delta}{d t^3} = Q + Q_1 \tan \delta + Q_2 \tan^2 \delta. \tag{46}$$

Introducing the constants for 1875.0 we have :

$$P = - \overset{s.}{0}.0000000290 + [1.2248167n] \sin \alpha \cos \alpha + [2.9401389] \cos^2 \alpha.$$
$$P_1 = + [0.0816772n] \cos \alpha + [2.8996767n] \sin \alpha + [2.8798278] \cos^2 \alpha \sin \alpha.$$
$$P_2 = - \overset{s.}{0}.0000000871 + [1.5258467n] \sin \alpha \cos \alpha + [3.2411689] \cos^2 \alpha.$$
$$P_3 = + [3.0047665] \cos^2 \alpha \sin \alpha + [2.4027065n] \sin \alpha. \tag{47}$$
$$Q = + [2.0359378] \sin \alpha + [4.0004500n] \cos \alpha + [3.2777678n] \sin^2 \alpha \cos \alpha.$$
$$Q_1 = + [2.4009130] \sin^2 \alpha + [4.1162302n] \sin \alpha \cos \alpha.$$
$$Q_2 = + [3.7548891n] \sin^2 \alpha \cos \alpha.$$

Tabular values of A log B, log C, A' log B', P, P_1, P_2, P_3, Q, Q_1, Q_2, are given in Publication XIV. of the Astronomische Gesellschaft, but they are not carried to a sufficient number of decimal places for our purpose.

The application of the formulæ given will now be illustrated by the computation of the differential coefficients for Groombridge 1119, for the epoch 1875.0.

$$\alpha = \overset{h.}{7}\ \overset{m.}{29}\ \overset{s.}{5}.631 \qquad \delta = \overset{\circ}{88}\ \overset{\prime}{59}\ \overset{\prime\prime}{37}.69$$

log sin α = 9.9663226	log cos α = 9.5786708n	log tan δ = 1.7553949
log sin 2 α = 9.8460234n	log cos 2 α = 9.8528923n	log tan² δ = 3.5107897
log sin 3 α = 9.5950715n	log cos 3 α = 9.9634456	log tan³ δ = 5.2661846
log sin 4 α = 9.9999457	log cos² α = 9.1573416	log tan⁴ δ = 7.0215795
log sin² α = 9.9326452	log cos³ α = 8.7360124n	log sec² δ = 3.5109236
log sin⁴ α = 9.8652904		

Computation of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$:

log n^s	0.1261147	log n''	1.3022060
log sin α	9.9663226	log cos α	9.5786708n
log tan δ	1.7553949	log n cos α	0.8808768n
log n sin α tan δ	1.8478322	$\frac{d\delta}{dt}$	$-7\overset{\prime\prime}{.}6011$
n sin α tan δ	+70.442079		
m	3.072245		
$\frac{d\alpha}{dt}$	$+73\overset{s.}{.}514324$		

Computation of $\frac{d^2\alpha}{dt^2}$:

By Bessel's Formulæ.

log $\tan^2\delta$	3.5107897
log $\sin 2\alpha$	9.8460234n
log constant	6.1138956
log (1)	9.4707087n
log $\tan\delta$	1.7553949
log $\cos\alpha$	9.5786708n
log constant	6.4752367
log (2)	7.8093024n
log $\sin 2\alpha$	9.8460234n
log constant	5.8128656
log (3)	5.6588890n
log $\tan\delta$	1.7553949
log $\sin\alpha$	9.9663226
log constant	4.7599145n
log (4)	6.4816320n

By Hill's Formulæ.

log $\frac{d\alpha}{dt}$	1.8663720
log constant	4.6338048n
log (1)	6.5001768n
log $\frac{d\alpha}{dt}$	1.8663720
log $\cos\alpha$	9.5786708n
log $\tan\delta$	1.7553949
log constant	5.9877809
log (2)	9.1882186n
log $\frac{d\delta}{dt}$	0.8808764n
log $\sin\alpha$	9.9663226
log $\sec^2\delta$	3.5109236
log constant	4.8116896
log (3)	9.1698122n

By Menten's Formulæ.

log $\sin 2\alpha$	9.8460234n	(1)	−.000053239
log constant	5.8128656	(2)	−.000132161
log (1)	5.6588890n	B	−.000185400
constant	+.00001899	log B	6.0738649n
(1)	−.00004559	log $\tan\delta$	1.7553949
A	−.00002660	log $B\tan\delta$	7.8292598n
log $\sin\alpha$	9.9663226	log $\sin 2\alpha$	9.8460234n
log constant	4.7599145n	log constant	6.1138956
log (1)	4.7262371n	log C	5.9599190n
log $\cos\alpha$	9.5786708n	log $\tan^2\delta$	3.5107897
log constant	6.4752367	log $C\tan^2\delta$	9.4707087n
log (2)	6.0539075n		

	s.
Constant	+0.00001899
(1)	− .29560297
(2)	− .00644618
(3)	− .00004559
(4)	− .00030313
$\frac{d^2\alpha}{dt^2}$	−0.30237888

	s.
Constant	+0.00003221
(1)	− .00031635
(2)	− .15424766
(3)	− .14784689
$\frac{d^2\alpha}{dt^2}$	−0.30237869

	s.
A	−0.00002660
$B\tan\delta$	− .00674931
$C\tan^2\delta$	− .29560297
$\frac{d^2\alpha}{dt^2}$	−0.30237888

Computation of $\frac{d^2\delta}{dt^2}$:

log $\tan\delta$	1.7553949
log $\sin^2\alpha$	9.9326452
log constant	7.2899869n
log (1)	8.9780270n
log $\sin\alpha$	9.9663226
log constant	7.6513280n
log (2)	7.6176506n
log $\cos\alpha$	9.5786708n
log constant	5.9360108n
log (3)	5.5146816

log $\frac{d\delta}{dt}$	0.8808764n
log constant	4.6338048n
log (1)	5.5146812
log $\frac{d\alpha}{dt}$	1.8663720
log $\sin\alpha$	9.9663226
log constant	7.1638722n
log (2)	8.9965668n

log $\sin\alpha$	9.9663226
log constant	7.6513280n
log (1)	7.6176506n
log $\cos\alpha$	9.5786708n
log constant	5.9360108n
log (2)	5.5146816
(1)	− .004146
(2)	+ .000033
A'	− .004113
log $\sin^2\alpha$	9.9326452
log constant	7.2899869n
log B'	7.2226321n
log $\tan\delta$	1.7553949
log $B'\tan\delta$	8.9780270n

	s.
(1)	−0.095066
(2)	− .004146
(3)	+ .000033
$\frac{d^2\delta}{dt^2}$	−0.099179

	″
(1)	+0.000033
(2)	− .099212
$\frac{d^2\delta}{dt^2}$	−0.099179

	″
A'	−0.004113
$B'\tan\delta$	− .095066
$\frac{d^2\delta}{dt^2}$	−0.099179

Computation of $\dfrac{d^3\alpha}{dt^3}$:

By Bessel's Formulæ.

log tan$^3\delta$	5.2661846
log sin 3α	9.5950715n
log constant	2.4027065
log (1)	7.2639626n
log tan$^2\delta$	3.5107897
log cos 2α	9.8528923n
log constant	2.9401389
log (2)	6.3038209n
log tan δ	1.7553949
log sin 3α	9.5950715n
log constant	2.2777678
log (3)	3.6282342n
log tan δ	1.7553949
log sin α	9.9663226
log constant	2.7811577n
log (4)	4.5028752n
log cos 2α	9.8528923n
log constant	2.6391089
log (5)	2.4920012n
log tan δ	1.7553949
log cos α	9.5786708n
log constant	0.8598285n
log (6)	2.1938942
log tan$^2\delta$	3.5107897
log sin 2α	9.8460234n
log constant	1.2248167n
log (7)	4.5816298
log sin 2α	9.8460234n
log constant	0.9237867
log (8)	0.7698101n

By Hill's Formulæ.

log cos 2α	9.8528923n
log constant	2.6391089
log (1)	2.4920012n
log sin 2α	9.8460234n
log constant	0.9237867n
log (2)	0.7698101
log sin α	9.9663226
log tan δ	1.7553949
log constant	2.6176279n
log (3)	4.3393454n
log sin α	9.9663226
log cos 2α	9.8528923n
log tan δ	1.7553949
log constant	2.5787978
log (4)	4.1534076n
log cos α	9.5786708n
log tan δ	1.7553949
log constant	0.8598285n
log (5)	2.1938942
log cos 2α	9.8528923n
log tan$^2\delta$	3.5107897
log constant	2.9401389
log (6)	6.3038209n
log sin 2α	9.8460234n
log tan$^2\delta$	3.5107897
log constant	1.2248167n
log (7)	4.5816298
log sin α	9.9663226
log tan$^3\delta$	5.2661846
log constant	2.4027065
log (8)	7.6352137
log sin α	9.9663226
log cos 2α	9.8528923n
log tan$^3\delta$	5.2661846
log constant	2.7037365
log (9)	7.7891360n

By Tiele's Formulæ.

log sin α		9.9663226
log cos α		9.5786708n
log constant		1.2248167n
log (1)		0.7698101
log cos$^2\alpha$		9.1573416
log constant		2.9401389
log (2)		2.0974805
constant	−	.0000000290
(1)	+	59
(2)	+	125
P	−	.0000000106
log cos α		9.5786708n
log constant		0.0816772n
log (3)		9.6603480
log sin α		9.9663226
log constant		2.8996767n
log (4)		2.8659993n
log cos$^2\alpha$		9.1573416
log sin α		9.9663226
log constant		2.8798278
log (5)		2.0034920
(3)	+	.000000000046
(4)	−	73451
(5)	+	10081
P_1	−	.000000063324
log P_1		2.8015683n
log tan δ		1.7553949
log P_1 tan δ		4.5569632n

log sin α		9.9663226
log cos α		9.5786708n
log constant		1.5258467n
log (6)		1.0708401
log cos$^2\alpha$		9.1573416
log constant		3.2411689
log (7)		2.3985105
constant	−	.00000008712
(6)	+	118
(7)	+	2503
P_2	−	.00000006091
log P_2		2.7846886n
log tan$^2\delta$		3.5107897
log P_2 tan$^2\delta$		6.2954783n
log cos$^2\alpha$		9.1573416
log sin α		9.9663226
log constant		3.0047665
log (8)		2.1284307
log sin α		9.9663226
log constant		2.4027065n
log (9)		2.3690291n
(8)	+	.000000013441
(9)	−	.000000023390
P_3	−	.000000009949
log P_3		1.9977794n
log tan$^3\delta$		5.2661846
log P_3 tan$^3\delta$		7.2639640n

Bessel:

		s.
Constant	+	0.0000000145
(1)	−	18363802
(2)	−	2012894
(3)	−	4248
(4)	−	31833
(5)	−	310
(6)	+	156
(7)	+	38162
(8)	+	6
$\dfrac{d^3\alpha}{dt^3}$	−	0.0020374618

Hill:

		s.
Constant	+	0.0000000145
(1)	−	310
(2)	+	6
(3)	−	21844
(4)	−	14237
(5)	+	156
(6)	−	2012894
(7)	+	38162
(8)	+	43173149
(9)	−	61536957
	−	0.0020374624

Tiele:

		s.
P	−	0.0000000106
P_1 tan δ	−	36055
P_2 tan$^2\delta$	−	1974596
P_3 tan$^3\delta$	−	18363861
	−	0.0020374618

Computation of $\frac{d^3\delta}{dt^3}$:

By Bessel's Formulæ.

log $\tan^2\delta$	3.5107897
log $\sin^2\alpha$	9.9326452
log $\cos\alpha$	9.5786708*n*
log constant	3.7548891*n*
log (1)	6.7769948
log $\tan\delta$	1.7553949
log $\sin\alpha$	9.9663226
log $\cos\alpha$	9.5786708*n*
log constant	4.1162302*n*
log (2)	5.4166185
log $\cos\alpha$	9.5786708*n*
log constant	4.0757367*n*
log (3)	3.6544075
log $\cos^3\alpha$	8.7360124*n*
log constant	3.2777678
log (4)	2.0137802*n*
log $\sin\alpha$	9.9663226
log constant	2.0359378
log (5)	2.0022604
log $\sin^2\alpha$	9.9326452
log $\tan\delta$	1.7553949
log constant	2.4009130
log (6)	4.0889531

By Hill's Formulæ.

log $\sin\alpha$	9.9663226
log constant	2.0359378
log (1)	2.0022604
log $\cos\alpha$	9.5786708*n*
log constant	4.0004500*n*
log (2)	3.5791208
log $\sin^2\alpha$	9.9326452
log $\cos\alpha$	9.5786708*n*
log constant	3.2777678*n*
log (3)	2.7890838
log $\sin^2\alpha$	9.8460234*n*
log $\tan\delta$	1.7553949
log constant	3.8152002*n*
log (4)	5.4166185
log $\sin^2\alpha$	9.9326452
log $\tan\delta$	1.7553949
log constant	2.4009130
log (5)	4.0889531
log $\sin^2\alpha$	9.9326452
log $\cos\alpha$	9.5786708*n*
log $\tan^2\delta$	3.5107897
log constant	3.7548891*n*
log (6)	6.7769948

By Tiele's Formulæ.

log $\sin\alpha$	9.9663226
log constant	2.0359378
log (1)	2.0022604
log $\cos\alpha$	9.5786708*n*
log constant	4.0004500*n*
log (2)	3.5791208
log $\sin^2\alpha$	9.9326452
log $\cos\alpha$	9.5786708*n*
log constant	3.2777678*n*
log (3)	2.7890838
(1)	+ .000000010
(2)	+ 379
(3)	+ 62
Q	+ .000000451

log $\sin^2\alpha$	9.9326452
log constant	2.4009130
log (4)	2.3335582
log $\sin\alpha$	9.9663226
log $\cos\alpha$	9.5786708*n*
log constant	4.1162302*n*
log (5)	3.6612236

(4)	+ .000000021555
(5)	+ 458378
Q_1	+ .000000479933

log Q_1	3.6811806
log $\tan\delta$	1.7553949
log $Q_1\tan\delta$	5.4365755
log $\sin^2\alpha$	9.9326452
log $\cos\alpha$	9.5786708*n*
log constant	3.7548891*n*
log Q_2	3.2662051*n*
log $\tan^2\delta$	3.5107897
log $Q_2\tan^2\delta$	6.7769948

		$''$
(1)	+	0.00059840
(2)	+	2610
(3)	+	45
(4)	−	1
(5)	+	1
(6)	+	123
$\frac{d^3\delta}{dt^3}$	+	0.00062618

		$''$
(1)	+	0.00000001
(2)	+	38
(3)	+	6
(4)	+	2610
(5)	+	123
(6)	+	59840
$\frac{d^3\delta}{dt^3}$	+	0.00062618

		$''$
Q	+	0.00000045
$Q_1\tan\delta$	+	2733
$Q_2\tan^2\delta$	+	59840
$\frac{d^3\delta}{dt^3}$	+	0.00062618

Computation of $\frac{d^4\alpha}{dt^4}$ and $\frac{d^4\delta}{dt^4}$:

By Bessel's Formulæ.

$\frac{d^4\alpha}{dt^4}$

log $\tan^4\delta$	7.0215795
log sin 4 α	9.9999457
log constant	$\bar{8}$.8676086
log (1)	5.8891338
log $\tan^3\delta$	5.2661846
log cos 3 α	9.9634456
log constant	$\bar{9}$.5299797
log (2)	4.7596099
log $\tan^2\delta$	3.5107897
log sin 4 α	9.9999457
log constant	$\bar{8}$.8676086
log (3)	2.3783440
log $\tan^2\delta$	3.5107897
log sin 2 α	9.8460234n
log constant	$\bar{9}$.6330799n
log (4)	2.9898930
log tan δ	1.7553949
log cos 3 α	9.9634456
log constant	$\bar{9}$.4050410
log (5)	1.1238815
log tan δ	1.7553949
log cos α	9.5786708n
log constant	$\bar{8}$.9667501n
log (6)	0.3008158
log sin 4 α	9.9999457
log constant	$\bar{7}$.9645186
log (7)	$\bar{7}$.9644643
log sin 2 α	9.8460234n
log constant	$\bar{9}$.3320498n
log (8)	$\bar{9}$.1780732

$\frac{d^4\delta}{dt^4}$

log $\tan^3\delta$	5.2661846
log $\sin^4\alpha$	9.8652904
log constant	$\bar{9}$.7426700
log (1)	4.8741450
log $\tan^3\delta$	5.2661846
log $\sin^2\alpha$	9.9326452
log $\cos^2\alpha$	9.1573416
log constant	0.3447299n
log (2)	4.7009013n
log $\tan^2\delta$	3.5107897
log sin α	9.9663226
log constant	0.4050411
log (3)	3.8821534
log $\tan^2\delta$	3.5107897
log sin α	9.9663226
log $\cos^2\alpha$	9.1573416
log constant	0.8821623
log (4)	3.5166162
log tan δ	1.7553949
log $\sin^2\alpha$	9.9326452
log constant	0.8333289
log (5)	2.5213690
log tan δ	1.7553949
log constant	0.4653522n
log (6)	2.2207471n
log tan δ	1.7553949
log $\sin^4\alpha$	9.8652904
log constant	$\bar{9}$.2655487
log (7)	0.8862340
log tan δ	1.7553949
log $\sin^2\alpha$	9.9326452
log $\cos^2\alpha$	9.1573416
log constant	0.1686387n
log (8)	1.0140204n
log sin α	9.9663226
log constant	0.4248824
log (9)	0.3912050
log sin α	9.9663226
log $\cos^2\alpha$	9.1573416
log constant	$\bar{0}$.4050411n
log (10)	$\bar{9}$.5287053n

		s.
(1)	+	0.00007747004
(2)	+	574923
(3)	+	2389
(4)	+	9770
(5)	+	133
(6)	+	20
(7)	+	0
(8)	+	1

$$\frac{d^4\alpha}{dt^4} = +0.00008334240$$

		″
(1)	+	0.0000074842
(2)	−	50223
(3)	+	7624
(4)	+	3286
(5)	+	332
(6)	−	166
(7)	+	8
(8)	−	10
(9)	+	2
(10)	−	0

$$\frac{d^4\delta}{dt^4} = +0.0000035695$$

It will be found convenient to represent the entire coefficients of $(t'-t_0)$, $(t'-t_0)^2$, $(t'-t_0)^3$, $(t'-t_0)^4$, &c., in equations (14) and (15) by a single expression.

Let

$$U^{\mathrm{I}} = \frac{d\,\alpha}{d\,t} \qquad \text{and} \qquad W^{\mathrm{I}} = \frac{d\,\delta}{d\,t}$$

$$U^{\mathrm{II}} = \frac{1}{2}\,\frac{d^2\,\alpha}{d\,t^2} \qquad W^{\mathrm{II}} = \frac{1}{2}\,\frac{d^2\,\delta}{d\,t^2}$$

$$U^{\mathrm{III}} = \frac{1}{2.3}\,\frac{d^3\alpha}{d\,t^3} \qquad W^{\mathrm{III}} = \frac{1}{2.3}\,\frac{d^3\,\delta}{d\,t^3}$$

$$U^{\mathrm{IV}} = \frac{1}{2.3.4}\,\frac{d^4\,\alpha}{d\,t^4} \qquad W^{\mathrm{IV}} = \frac{1}{2.3.4}\,\frac{d^4\,\delta}{d\,t^4}$$

&c., &c.

Then writing t for $(t'-t_0)$ we have,

$$\alpha = \alpha_0 + U^{\mathrm{I}}\,t + U^{\mathrm{II}}\,t^2 + U^{\mathrm{III}}\,t^3 + U^{\mathrm{IV}}\,t^4, \text{ \&c.} \tag{48}$$

$$\delta = \delta_0 + W^{\mathrm{I}}\,t + W^{\mathrm{II}}\,t^2 + W^{\mathrm{III}}\,t^3 + W^{\mathrm{IV}}\,t^4, \text{ \&c.} \tag{49}$$

We now compute the right ascension of Groombridge 1119 for $t = -20, -40, -80, -120$, corresponding to the years 1855, 1835, 1795, and 1755.

With the data

$$\frac{d\,\alpha}{d\,t} = +73\overset{s.}{.}514324 \qquad \frac{d^2\,\alpha}{d\,t^2} = -\overset{s.}{.}30237888 \qquad \frac{d^3\,\alpha}{d\,t^3} = -\overset{s.}{.}0020374621 \qquad \frac{d^4\,\alpha}{d\,t^4} = +\overset{s.}{.}00008334240$$

and the values of

$$U^{\mathrm{I}} = 73\overset{s.}{.}514324 \qquad U^{\mathrm{II}} = -\overset{s.}{.}15118944 \qquad U^{\mathrm{III}} = -\overset{s.}{.}000339577 \qquad U^{\mathrm{IV}} = +\overset{s.}{.}0000034726$$

we have,

	For 1855.	For 1835.	For 1795.	For 1755.
	h m. s.	h. m. s.	h. m. s.	h. m. s.
α_0	= 7 29 5.6310	7 29 5.6310	7 29 5.6310	7 29 5.6310
$U^{\mathrm{I}}\ t$	− 24 30.2865	− 49 0.5730	−1 38 1.1459	−2 27 1.7189
$U^{\mathrm{II}}\ t^2$	− 1 0.4758	− 4 1.9031	− 16 7.6124	− 36 17.1279
$U^{\mathrm{III}}\ t^3$	+ 2.7166	+ 21.7329	+ 2 53.8634	+ 9 46.7890
$U^{\mathrm{IV}}\ t^4$	+ 0.5556	+ 8.8899	+ 2 22.2377	+ 12 0.0783
α	7 3 38.1409	6 36 33.7778	5 40 12.9738	4 47 33.6515

The corresponding values of α obtained from equations (6) are:

h. m. s.	h. m. s.	h. m. s.	h. m. s.
7 3 38.153	6 36 34.034	5 40 10.397	4 45 55.203

giving the deviations

$+0\overset{s.}{.}012$	$+0\overset{s.}{.}256$	$-2\overset{s.}{.}577$	$-98\overset{s.}{.}449$

It will be seen from this example, that, at the end of twenty years, equation (48) to the third term inclusive fails by $0^s.56$; but that when the fourth term is included, the place by equation (48) is reproduced within about $0^s.01$. For any time greater than twenty years, terms involving higher powers of t will be required in order to obtain a correspondence with the results given by equations (6).

Computation of the Differential Coefficients from Successive Orders of Differences of the given Functions.

Let Δ_1, Δ_2, Δ_3, &c., represent the successive orders of differences of the computed functions.

Let Δ^{I}, Δ^{II}, Δ^{III}, &c., represent the differences opposite the initial function.

For the odd differences,

$$\Delta^{\mathrm{I}} = \frac{\Delta_1^{+\frac{1}{2}} + \Delta_1^{-\frac{1}{2}}}{2}, \quad \Delta^{\mathrm{III}} = \frac{\Delta_3^{+\frac{1}{2}} + \Delta_3^{-\frac{1}{2}}}{2}, \quad \Delta^{\mathrm{V}} = \frac{\Delta_5^{+\frac{1}{2}} + \Delta_5^{-\frac{1}{2}}}{2}, \quad \&c.$$

For the even differences,

$$\Delta^{\mathrm{II}} = \Delta_2, \quad \Delta^{\mathrm{IV}} = \Delta_4, \quad \Delta^{\mathrm{VI}} = \Delta_6, \quad \&c.$$

Then (see Brünnow, Spher. Astron., 1865, p. 28),

$$\begin{aligned}
\frac{d\alpha}{dt} &= \frac{1}{w}\left[\Delta^{\mathrm{I}} - \tfrac{1}{6}\Delta^{\mathrm{III}} + \tfrac{1}{30}\Delta^{\mathrm{V}} - \tfrac{1}{140}\Delta^{\mathrm{VII}} + \tfrac{1}{630}\Delta^{\mathrm{IX}} - \&c.\right] \\
\frac{d^2\alpha}{dt^2} &= \frac{1}{w^2}\left[\Delta^{\mathrm{II}} - \tfrac{1}{12}\Delta^{\mathrm{IV}} + \tfrac{1}{90}\Delta^{\mathrm{VI}} - \tfrac{1}{560}\Delta^{\mathrm{VIII}} + \tfrac{1}{3150}\Delta^{\mathrm{X}} - \&c.\right] \\
\frac{d^3\alpha}{dt^3} &= \frac{1}{w^3}\left[\Delta^{\mathrm{III}} - \tfrac{1}{4}\Delta^{\mathrm{V}} + \tfrac{7}{120}\Delta^{\mathrm{VII}} - \tfrac{41}{3024}\Delta^{\mathrm{IX}} + \&c.\right] \\
\frac{d^4\alpha}{dt^4} &= \frac{1}{w^4}\left[\Delta^{\mathrm{IV}} - \tfrac{1}{6}\Delta^{\mathrm{VI}} + \tfrac{7}{240}\Delta^{\mathrm{VIII}} - \tfrac{41}{7560}\Delta^{\mathrm{X}} + \&c.\right] \\
\frac{d^5\alpha}{dt^5} &= \frac{1}{w^5}\left[\Delta^{\mathrm{V}} - \tfrac{1}{3}\Delta^{\mathrm{VII}} + \tfrac{13}{144}\Delta^{\mathrm{IX}} - \&c.\right] \\
\frac{d^6\alpha}{dt^6} &= \frac{1}{w^6}\left[\Delta^{\mathrm{VI}} - \tfrac{1}{4}\Delta^{\mathrm{VIII}} + \tfrac{13}{240}\Delta^{\mathrm{X}} - \&c.\right] \\
\frac{d^7\alpha}{dt^7} &= \frac{1}{w^7}\left[\Delta^{\mathrm{VII}} - \tfrac{5}{12}\Delta^{\mathrm{IX}} + \&c.\right] \\
\frac{d^8\alpha}{dt^8} &= \frac{1}{w^8}\left[\Delta^{\mathrm{VIII}} - \tfrac{1}{3}\Delta^{\mathrm{X}} + \&c.\right] \\
\frac{d^9\alpha}{dt^9} &= \frac{1}{w^9}\left[\Delta^{\mathrm{IX}} - \&c.\right] \\
\frac{d^{10}\alpha}{dt^{10}} &= \frac{1}{w^{10}}\left[\Delta^{\mathrm{X}} - \&c.\right]
\end{aligned} \tag{50}$$

The following cases occur in which this method of development is applicable:

Case	Given	From	To find
(a)	α	α	$\frac{d\alpha}{dt}$ $\frac{d^2\alpha}{dt^2}$ $\frac{d^3\alpha}{dt^3}$ $\frac{d^4\alpha}{dt^4}$ $\frac{d^5\alpha}{dt^5}$, &c.
(b)	α $\frac{d\alpha}{dt}$	$\frac{d\alpha}{dt}$	$\frac{d^2\alpha}{dt^2}$ $\frac{d^3\alpha}{dt^3}$ $\frac{d^4\alpha}{dt^4}$ $\frac{d^5\alpha}{dt^5}$ $\frac{d^6\alpha}{dt^6}$, &c.
(c)	α $\frac{d\alpha}{dt}$ $\frac{d^2\alpha}{dt}$	$\frac{d^2\alpha}{dt^2}$	$\frac{d^3\alpha}{dt^3}$ $\frac{d^4\alpha}{dt^4}$ $\frac{d^5\alpha}{dt^5}$ $\frac{d^6\alpha}{dt^6}$ $\frac{d^7\alpha}{dt^7}$, &c.
(d)	α $\frac{d\alpha}{dt}$ $\frac{d^2\alpha}{dt^2}$ $\frac{d^3\alpha}{dt^3}$	$\frac{d^3\alpha}{dt^3}$	$\frac{d^4\alpha}{dt^4}$ $\frac{d^5\alpha}{dt^5}$ $\frac{d^6\alpha}{dt^6}$ $\frac{d^7\alpha}{dt^7}$ $\frac{d^8\alpha}{dt^8}$, &c.
(e)	α $\frac{d\alpha}{dt}$ $\frac{d^2\alpha}{dt^2}$ $\frac{d^3\alpha}{dt^3}$ $\frac{d^4\alpha}{dt^4}$	$\frac{d^4\alpha}{dt^4}$	$\frac{d^5\alpha}{dt^5}$ $\frac{d^6\alpha}{dt^6}$ $\frac{d^7\alpha}{dt^7}$ $\frac{d^8\alpha}{dt^8}$ $\frac{d^9\alpha}{dt^9}$, &c.

and similarly for δ, $\frac{d\delta}{dt}$, $\frac{d^2\delta}{dt^2}$, &c.

For the application of this method in the computation of the differential coefficients of Groombridge 1119 we have the following data, for 1, 8, and 16 years.

CASE (a). INITIAL FUNCTION $= \alpha$.

FOR $w = 1$.

	h.	*m.*	*s.*	Δ_1	Δ_2	Δ_3	Δ_4	
1871.0	7	24	9.177					
				+ 1 14.560				
1872.0	7	25	23.737		− .297			
				+ 1 14.263		+ .000		
1873.0	7	26	38.000		− .297		− .004	
				+ 1 13.966		− .004		
1874.0	7	27	51.966		− .301		+ .003	
				+ 1 13.665		− .001		
1875.0	7	29	5.631		− .302		− .002	+ .000
				+ 1 13.363		− .003		
1876.0	7	30	18.994		− .305		+ .002	
				+ 1 13.058		− .001		
1877.0	7	31	32.052		− .306		− .001	
				+ 1 12.752		− .002		
1878.0	7	32	44.804		− .308			
				+ 1 12.444				
1879.0	7	33	57.248					

CASE (a). INITIAL FUNCTION $= \alpha$.

FOR $w = 8$.

	h.	m.	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	
1827.0	6	25	25.071								
				+11 8.963							
1835.0	6	36	34.034		− 9.772						
				+10 59.191		−2.571					
1843.0	6	47	33.225		−12.343		+.308				
				+10 46.848		−2.263		+.030			
1851.0	6	58	20.073		−14.606		+.338		−.018		
				+10 32.242		−1.925		+.012		+.014	
1859.0	7	8	52.315		−16.531		+.350		−.004		
				+10 15.711		−1.575		+.008		−.023	
1867.0	7	19	8.026		−18.106		+.358		−.027		
				+ 9 57.605		−1.217		−.019		+.020	
1875.0	7	29	5.631		−19.323		+.339		−.007		+.002
				+ 9 38.282		− .878		−.026		+.006	
1883.0	7	38	43.913		−20.201		+.313		−.001		
				+ 9 18.081		− .565		−.027		−.009	
1891.0	7	48	1.994		−20.766		+.286		−.010		
				+ 8 57.315		− .279		−.037		+.007	
1899.0	7	56	59.309		−21.045		+.249		−.003		
				+ 8 36.270		− .030		−.040			
1907.0	8	5	35.579		−21.075		+.209				
				+ 8 15.195		+ .179					
1915.0	8	13	50.774		−20.896						
				+ 7 54.299							
1923.0	8	21	45.073								

CASE (a). INITIAL FUNCTION $= \alpha$.

FOR $w = 16$.

	h.	m.	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}
1795.0	5	40	10.397										
				+22 38.936									
1811.0	6	2	49.333		− 3.198								
				+22 35.738		−24.386							
1827.0	6	25	25.071		−27.584		+2.906						
				+22 8.154		−21.480		+1.864					
1843.0	6	47	33.225		−49.064		+4.770		−1.103				
				+21 19.090		−16.710		+ .761		+.165			
1859.0	7	8	52.315		−65.774		+5.531		− .938		+.108		
				+20 13.316		−11.179		− .177		+.273		−.022	
1875.0	7	29	5.631		−76.953		+5.354		− .665		+.086		−.094
				+18 56.363		− 5.825		− .842		+.359		−.116	
1891.0	7	48	1.994		−82.778		+4.512		− .306		−.030		
				+17 33.585		− 1.313		−1.148		+.329			
1907.0	8	5	35.579		−84.091		+3.364		+ .023				
				+16 9.494		+ 2.051		−1.125					
1923.0	8	21	45.073		−82.040		+2.239						
				+14 47.454		+ 4.290							
1939.0	8	36	32.527		−77.750								
				+13 29.704									
1955.0	8	50	2.231										

CASE (*b*). INITIAL FUNCTION $= \dfrac{d\alpha}{dt}$.

FOR $w = 1$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	
1871.0	+ 74.705491					
		− 294443				
1872.0	+ 74.411048		− 2294			
		− 296737		+ 97		
1873.0	+ 74.114311		− 2197		− 9	
		− 298934		+ 78		
1874.0	+ 73.815377		− 2119		− 1	
		− 301053		+ 77		
1875.0	+ 73.514324		− 2042		+ 12	+ 0
		− 303095		+ 89		
1876.6	+ 73.211229		− 1953		− 8	
		− 305048		+ 81		
1877.0	+ 72.906181		− 1872		+ 6	
		− 306920		+ 87		
1878.0	+ 72.599261		− 1785			
		− 308705				
1879.0	+ 72.290556					

CASE (*b*). INITIAL FUNCTION $= \dfrac{d\alpha}{dt}$.

FOR $w = 8$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1843.0	+ 81.667645								
		− 1.688364							
1851.0	+ 79.979281		− .262517						
		− 1.950881		+ 43548					
1859.0	+ 78.028400		− .218969		+ 1045				
		− 2.169850		+ 44593		− 2037			
1867.0	+ 75.858550		− .174376		− 992		+ 514		
		− 2.344226		+ 43601		− 1523		− 20	
1875.0	+ 73.514324		− .130775		− 2515		+ 494		− 135
		− 2.475001		+ 41086		− 1029		− 155	
1883.0	+ 71.039323		− .089689		− 3544		+ 339		
		− 2.564690		+ 37542		− 690			
1891.0	+ 68.474633		− .052147		− 4234				
		− 2.616837		+ 33308					
1899.0	+ 65.857796		− .018839						
		− 2.635676							
1907.0	+ 63.222120								

CASE (*b*). INITIAL FUNCTION $= \dfrac{d\alpha}{dt}$.

FOR $w = 16$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1811.0	+85.071302								
		−0.972057							
1827.0	+84.099245		−1.459543						
		−2.431600		+.251898					
1843.0	+81.667645		−1.207645		+80916				
		−3.639245		+.332814		−64514			
1859.0	+78.028400		− .874831		+16402		+11689		
		−4.514076		+.349216		−52825		+11973	
1875.0	+73.514324		− .525615		−36423		+23662		−13890
		−5.039691		+.312793		−29163		− 1917	
1891.0	+68.474633		− .212822		−65586		+21745		
		−5.252513		+.247207		− 7418			
1907.0	+63.222120		+ .034385		−73004				
		−5.218128		+.174203					
1923.0	+58.003992		+ .208588						
		−5.009540							
1939.0	+52.994452								

CASE (*c*). INITIAL FUNCTION $= \dfrac{d^2\alpha}{dt^2}$.

FOR $w = 1$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	
1871.0	− 0.29354576					
		− 233529				
1872.0	− 0.29880105		+ 8503			
		− 225026		− 53		
1873.0	− 0.29813131		+ 8450		− 2	
		− 216576		− 55		
1874.0	− 0.30029707		+ 8395		− 1	
		− 208181		− 56		
1875.0	− 0.30237888		+ 8339		− 11	− 4
		− 199842		− 67		
1876.0	− 0.30437730		+ 8272		− 7	
		− 191570		− 74		
1877.0	− 0.30629300		+ 8198		+ 0	
		− 183372		− 74		
1878.0	− 0.30812672		+ 8124			
		− 175248				
1879.0	− 0.30987920					

CASE (c). INITIAL FUNCTION $= \dfrac{d^2\alpha}{dt^2}$.

FOR $w = 8$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1843.0	− 0.19318438								
		− 3544774							
1851.0	− 0.22863212		+ 531344						
		− 3013430		+ 24218					
1859.0	− 0.25876642		+ 555562		− 25290				
		− 2457868		− 1072		+ 4005			
1867.0	− 0.28334510		+ 554490		− 21285		+ 947		
		− 1903378		− 22357		+ 4952		− 782	
1875.0	− 0.30237888		+ 532133		− 16333		+ 165		+ 274
		− 1371245		− 38690		+ 5117		− 508	
1883.0	− 0.31609133		+ 493443		− 11216		− 343		
		− 877802		− 49906		+ 4774			
1891.0	− 0.32486935		+ 443537		− 6442				
		− 434265		− 56348					
1899.0	− 0.32921200		+ 387189						
		− 47076							
1907.0	− 0.32968276								

CASE (c). INITIAL FUNCTION $= \dfrac{d^2\alpha}{dt^2}$.

FOR $w = 16$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1811.0	−0.01260433								
		−9609612							
1827.0	−0.10870045		+1161219						
		−8448393		+728970					
1843.0	−0.19318438		+1890189		−422201				
		−6558204		+306769		+ 30673			
1859.0	−0.25876642		+2196958		−391528		+101121		
		−4361246		− 84759		+131794		−80033	
1875.0	−0.30237888		+2112199		−259734		+ 20088		+24336
		−2249047		−344493		+151882		−55697	
1891.0	−0.32486935		+1767706		−107852		− 35609		
		− 481341		−452345		+116273			
1907.0	−0.32968276		+1315361		+ 8421				
		+ 834020		−443924					
1923.0	−0.32134256		+ 871437						
		+1705457							
1939.0	−0.30428799								

CASE (d). INITIAL FUNCTION $= \dfrac{d^3 \alpha}{d t^3}$.

FOR $w = 1$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	
1871.0	− 0.0023754634					
		+ 852882				
1872.0	− 0.0022901752		− 4971			
		+ 847911		− 425		
1873.0	− 0.0022053841		− 5396		+ 11	
		+ 842515		− 414		
1874.0	− 0.0021211326		− 5810		+ 13	
		+ 836705		− 401		
1875.0	− 0.0020374621		− 6211		+ 12	+ 12
		+ 830494		− 389		
1876.0	− 0.0019544127		− 6600		+ 11	
		+ 823894		− 378		
1877.0	− 0.0018720233		− 6978		+ 13	
		+ 816916		− 365		
1878.0	− 0.0017903317		− 7343			
		+ 809573				
1879.0	− 0.0017093744					

CASE (d). INITIAL FUNCTION $= \dfrac{d^3 \alpha}{d t^3}$.

FOR $w = 8$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1843.0	−0.0047421960								
		+6377656							
1851.0	−0.0041044304		+472678						
		+6850334		−335216					
1859.0	−0.0034193970		+137462		+41511				
		+6987796		−293705		+16005			
1867.0	−0.0027206174		−156243		+57516		−9284		
		+6831553		−236189		+ 6721		+1213	
1875.0	−0.0020374621		−392432		+64237		−8071		+1823
		+6439121		−171952		− 1350		+3036	
1883.0	−0.0013935500		−564384		+62887		−5035		
		+5874737		−109065		− 6385			
1891.0	−0.0008060763		−673449		+56502				
		+5201388		− 52563					
1899.0	−0.0002859375		−726012						
		+4475376							
1907.0	+0.0001616001								

CASE (d). INITIAL FUNCTION $= \dfrac{d^3 \alpha}{d t^3}$.

FOR $w = 16$.

	$s.$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1811.0	−0.0061613944								
		+ 4345993							
1827.0	−0.0057367951		+5599998						
		+ 9945991		−2317999					
1843.0	−0.0047421960		+3281999		−372641				
		+13227990		−2690640		+966431			
1859.0	−0.0034193970		+ 591359		+593790		−594974		
		+13819349		−2096850		+371457		+132217	
1875.0	−0.0020374621		−1505491		+965247		−462757		+110012
		+12313858		−1131603		− 91300		+242229	
1891.0	−0.0008060763		−2637094		+873947		−220528		
		+ 9676764		− 257656		−311828			
1907.0	+0.0001616001		−2894750		+562119				
		+ 6782014		+ 304463					
1923.0	+0.0008398015		−2590287						
		+ 4191727							
1939.0	+0.0012589742								

CASE (e). INITIAL FUNCTION $= \dfrac{d^4 \alpha}{d t^4}$.

FOR $w = 1$.

	$s.$	Δ_1	Δ_2	Δ_3	Δ_4	
1871.0	+ 0.00008548692					
		− 47054				
1872.0	+ 0.00008501638		− 4482			
		− 51536		+ 165		
1873.0	+ 0.00008450102		− 4317		− 4	
		− 55853		+ 161		
1874.0	+ 0.00008394249		− 4156		− 2	
		− 60009		+ 159		
1875.0	+ 0.00008334240		− 3997		− 1	+ 1
		− 64006		+ 158		
1876.0	+ 0.00008270234		− 3839		+ 3	
		− 67845		+ 161		
1877.0	+ 0.00008202389		− 3678		+ 7	
		− 71523		+ 168		
1878.0	+ 0.00008130866		− 3510			
		− 75033				
1879.0	+ 0.00008055833					

CASE (*e*). INITIAL FUNCTION $= \dfrac{d^4\alpha}{dt^4}$.

FOR $w = 8$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1843.0	+0.00007522783								
		+811510							
1851.0	+0.00008334293		−435399						
		+376111		+37978					
1859.0	+0.00008710404		−397421		+25899				
		− 21310		+63877		−11902			
1867.0	+0.00008689094		−333544		+13997		+ 950		
		−354854		+77874		−10952		+1496	
1875.0	+0.00008334240		−255670		+ 3045		+2446		−898
		−610524		+80919		− 8506		+ 598	
1883.0	+0.00007723716		−174751		− 5461		+3004		
		−785275		+75458		− 5502			
1891.0	+0.00006938441		− 99293		−10963				
		−884568		+64495					
1899.0	+0.00006053873		− 34798						
		−919366							
1907.0	+0.00005134507								

CASE (*e*). INITIAL FUNCTION $= \dfrac{d^4\alpha}{dt^4}$.

FOR $w = 16$.

	s.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
1811.0	+0.00000543240								
		+4088210							
1827.0	+0.00004631450		−1196877						
		+2891333		−506835					
1843.0	+0.00007522783		−1703712		+646762				
		+1187621		+139927		−242539			
1859.0	+0.00008710404		−1563785		+404223		− 94334		
		− 376164		+544150		−336873		+210497	
1875.0	+0.00008334240		−1019635		+ 67350		+116163		−165015
		−1395799		+611500		−220710		+ 45482	
1891.0	+0.00006938441		− 408135		−153360		+161645		
		−1803934		+458140		− 59065			
1907.0	+0.00005134507		+ 50005		−212425				
		−1753929		+245715					
1923.0	+0.00003380578		+ 295720						
		−1458209							
1939.0	+0.00001922369								

For an illustration of the application of formulæ (50), and the subsequent computation of the terms U^{II}, U^{III}, &c., we select case (*b*) and case (*e*), in each of which $w = 16$.

CASE (*b*).

	s.
Δ^{I}	-4.776884
$-\frac{1}{6}\,\Delta^{\text{III}}$	-0.055167
$+\frac{1}{30}\,\Delta^{\text{V}}$	-0.001366
$-\frac{1}{140}\,\Delta^{\text{VII}}$	-0.000036
sum	-4.833453
log sum	$0.6842575n$
log 16	1.2041200
$\log \frac{d^2\alpha}{dt^2}$	$9.4801375n$
log 2	0.3010300
$\log U^{\text{II}}$	$9.1791075n$
U^{II}	-0.151045

	s.
Δ^{II}	-0.525615
$-\frac{1}{12}\,\Delta^{\text{IV}}$	$+0.003035$
$+\frac{1}{90}\,\Delta^{\text{VI}}$	$+0.000263$
$-\frac{1}{560}\,\Delta^{\text{VIII}}$	$+0.000025$
sum	-0.522292
log sum	$9.7179134n$
$\log 16^2$	2.4082400
$\log \frac{d^3\alpha}{dt^3}$	$7.3096734n$
log 6	0.7781513
$\log U^{\text{III}}$	$6.5315221n$
U^{III}	-0.00034003

	s.
Δ^{III}	$+0.331004$
$-\frac{1}{4}\,\Delta^{\text{V}}$	$+0.010248$
$+\frac{7}{120}\,\Delta^{\text{VII}}$	$+0.000293$
sum	$+0.341545$
log sum	9.53345
$\log 16^3$	3.61236
$\log \frac{d^4\alpha}{dt^4}$	5.92109
log 24	1.38021
$\log U^{\text{IV}}$	4.54088
U^{IV}	$+0.0000034744$

	s.
Δ^{IV}	-0.036423
$-\frac{1}{6}\,\Delta^{\text{VI}}$	-0.003944
$+\frac{7}{240}\,\Delta^{\text{VIII}}$	-0.000405
sum	-0.040772
log sum	$8.61036n$
$\log 16^4$	4.81648
$\log \frac{d^5\alpha}{dt^5}$	$3.79388n$
log 120	2.07918
$\log U^{\text{V}}$	$1.71470n$
U^{V}	-0.000000005184

	s.
Δ^{V}	-0.040994
$-\frac{1}{3}\,\Delta^{\text{VII}}$	-0.001676
sum	-0.042670
log sum	$8.63012n$
$\log 16^5$	6.02060
$\log \frac{d^6\alpha}{dt^6}$	$2.60952n$
log 720	2.85733
$\log U^{\text{VI}}$	$\bar{9}.75219n$
U^{VI}	-0.00000000005652

	s.
Δ^{VI}	$+0.023662$
$-\frac{1}{4}\,\Delta^{\text{VIII}}$	$+0.003472$
sum	$+0.027134$
log sum	8.43351
$\log 16^6$	7.22472
$\log \frac{d^7\alpha}{dt^7}$	1.20879
log 5040	3.70243
$\log U^{\text{VII}}$	$\bar{7}.50636$
U^{VII}	$+0.0000000000003209$

	s.
Δ^{VII}	+ 0.005028
sum	+ 0.005028
log sum	7.70140
log 16^7	8.42884
log $\frac{d^8 \alpha}{d t^8}$	9.27256
log 40320	4.60552
log U^{VIII}	$\bar{4}.66704$
U^{VIII}	+ 0.000000000000000464

	s.
Δ^{VIII}	− 0.013890
sum	− 0.013890
log sum	8.14270*n*
log 16^8	9.63296
log $\frac{d^9 \alpha}{d t^9}$	8.50974*n*
log 362880	5.55976
log U^{IX}	$\bar{2}.94998n$
U^{IX}	− 0.0000000000000000891

CASE (*e*).

	s.
Δ^{I}	− 0.00000885981
$-\frac{1}{6}\Delta^{III}$	− 96304
$+\frac{1}{30}\Delta^{V}$	− 9293
$-\frac{1}{140}\Delta^{VII}$	− 914
sum	− 0.00000992492
log sum	4.99673*n*
log 16	1.20412
log $\frac{d^5 \alpha}{d t^5}$	3.79261*n*
log 120	2.07918
log U^{V}	1.71343*n*
U^{V}	− 0.000000005169

	s.
Δ^{II}	− 0.00001019635
$-\frac{1}{12}\Delta^{IV}$	− 5612
$+\frac{1}{90}\Delta^{VI}$	+ 1291
$-\frac{1}{560}\Delta^{VIII}$	+ 294
sum	− 0.00001023662
log sum	5.01016*n*
log 16^2	2.40824
log $\frac{d^6 \alpha}{d t^6}$	2.60192*n*
log 720	2.85733
log U^{VI}	$\bar{9}.74459n$
U^{VI}	− 0.00000000005554

	s.
Δ^{III}	+ 0.00000577825
$-\frac{1}{4}\Delta^{V}$	+ 69698
$+\frac{7}{120}\Delta^{VII}$	+ 7466
sum	+ 0.00000654989
log sum	4.81623
log 16^3	3.61236
log $\frac{d^7 \alpha}{d t^7}$	1.20387
log 5040	3.70243
log U^{VII}	$\bar{7}.50140$
U^{VII}	+ 0.0000000000003173

	s.
Δ^{IV}	+ 0.00000067350
$-\frac{1}{6}\Delta^{VI}$	− 19360
$+\frac{7}{240}\Delta^{VIII}$	− 4813
sum	+ 0.00000043177
log sum	3.63525
log 16^4	4.81648
log $\frac{d^8 \alpha}{d t^8}$	8.81877
log 40320	4.60552
log U^{VIII}	$\bar{4}.21325$
U^{VIII}	+ 0.000000000000000163

	s.		s.
Δ^{V}	-0.00000278791	Δ^{VI}	$+0.00000116163$
$-\frac{1}{3}\Delta^{\mathrm{VII}}$	$-\qquad 42663$	$-\frac{1}{4}\Delta^{\mathrm{VIII}}$	$+\qquad 41254$
sum	-0.00000321454	sum	$+0.00000157417$
log sum	$4.50712n$	log sum	4.19706
$\log 16^5$	6.02060	$\log 16^6$	7.22472
$\log \frac{d^9\alpha}{dt^9}$	$8.48652n$	$\log \frac{d^{10}\alpha}{dt^{10}}$	6.97234
log 362880	5.55976	log 3628800	6.55976
$\log U^{\mathrm{IX}}$	$\bar{2}.92676n$	$\log U^{\mathrm{X}}$	$\bar{0}.41258$
	s.		s.
U^{IX}	$-0.0000000000000000000845$	U^{X}	$+0.000000000000000000000259$

	s.		s.
Δ^{VII}	$+0.00000127989$	Δ^{VIII}	-0.00000165015
sum	$+0.00000127989$	sum	-0.00000165015
log sum	4.10718	log sum	$4.21751n$
$\log 16^7$	8.42884	$\log 16^9$	9.63296
$\log \frac{d^{11}\alpha}{dt^{11}}$	5.67834	$\log \frac{d^{12}\alpha}{dt^{12}}$	$4.58455n$
log 39916800	7.60116	log 479001600	8.68034
$\log U^{\mathrm{XI}}$	$\bar{\bar{8}}.07718$	$\log U^{\mathrm{XII}}$	$\bar{\bar{5}}.90421n$
	s.		s.
U^{XI}	$+0.00000000000000000000000119$	U^{XII}	$-0.0000000000000000000000000080$

The values of U^{I}, U^{II}, . . . U^{XII}, given on the following pages, were obtained from the differential coefficients derived from equations (50), employing the data given on pp. 249–256.

In the values marked by an asterisk the differential coefficients were computed as follows: —

$\frac{d\alpha}{dt}$; Bessel's Formulæ.

$\frac{d^2\alpha}{dt^2}$; Menten.

$\frac{d^3\alpha}{dt^3}$; Mean of Bessel and Hill.

$\frac{d^4\alpha}{dt^4}$; Bessel.

Separate values of the Coefficients U^{I}, U^{II}, . . . U^{XII}.

Initial Function.	U^{I}	Logarithms.
	s.	
*	+73.514324	1.8663720
α for $w = 1$	+73.51433	1.8663720
$w = 8$	+73.51464	1.8663739
$w = 16$	+73.51481	1.8663749

Initial Function.	U^{II}	Logarithms.
	s.	
*	− 0.15118944	9.1795214*n*
α for $w = 1$	− 0.151000	9.1789769*n*
$w = 8$	− 0.151182	9.1795001*n*
$w = 16$	− 0.151185	9.1795087*n*
$\frac{d\alpha}{dt}$ for $w = 1$	− 0.151030	9.1780632*n*
$w = 8$	− 0.151045	9.1791064*n*
$w = 16$	− 0.151045	9.1791064*n*

Initial Function.	U^{III}	Logarithms.
	s.	
*	− 0.000339577	6.53094*n*
α for $w = 1$	− 333	6.52244*n*
$w = 8$	− 33912	6.53035*n*
$w = 16$	− 33998	6.53145*n*
$\frac{d\alpha}{dt}$ for $w = 1$	− 34044	6.53204*n*
$w = 8$	− 34000	6.53148*n*
$w = 16$	− 34003	6.53152*n*
$\frac{d^2\alpha}{dt^2}$ for $w = 1$	− 34000	6.53148*n*
$w = 8$	− 34001	6.53149*n*
$w = 16$	− 34002	6.53150*n*

U^{IV}

Initial Function.		s.	Logarithms
*	+	0.0000034726 †	4.54065
α for $w = 8$	+	34605	4.53914
$w = 16$	+	34761	4.54109
$\frac{d\alpha}{dt}$ for $w = 1$	+	34584	4.53888
$w = 8$	+	34714	4.54050
$w = 16$	+	34744	4.54088
$\frac{d^2\alpha}{dt^2}$ for $w = 1$	+	34746	4.54090
$w = 8$	+	34732	4.54073
$w = 16$	+	34734	4.54075
$\frac{d^3\alpha}{dt^3}$ for $w = 1$	+	34820	4.54183
$w = 8$	+	34729	4.54069
$w = 16$	+	34737	4.54079

U^{V}

		s.	
α for $w = 8$	−	0.000000005893	1.77034n
$w = 16$	−	4936	1.69338n
$\frac{d\alpha}{dt}$ for $w = 8$	−	5284	1.72296n
$w = 16$	−	5184	1.71466n
$\frac{d^2\alpha}{dt^2}$ for $w = 1$	−	5083	1.70612n
$w = 8$	−	5179	1.71424n
$w = 16$	−	5169	1.71341n
$\frac{d^3\alpha}{dt^3}$ for $w = 1$	−	5177	1.71408n
$w = 8$	−	5181	1.71441n
$w = 16$	−	5180	1.71433n
$\frac{d^4\alpha}{dt^4}$ for $w = 1$	−	5172	1.71366n
$w = 8$	−	5169	1.71341n
$w = 16$	−	5169	1.71341n

† With the coefficient 8, given by Engelmann, the value $+0^s.0000034726$ becomes $+0^s.0000033928$.

U^{VI}

Initial Function.			Logarithms.
		s.	
α for $w = 8$	$-$	0.00000000003709	$\bar{9}.56926n$
$w = 16$	$-$	5725	$\bar{9}.75778n$
$\frac{d\alpha}{dt}$ for $w = 8$	$-$	5281	$\bar{9}.72272n$
$w = 16$	$-$	5652	$\bar{9}.75220n$
$\frac{d^2\alpha}{dt^2}$ for $w = 1$	$-$	5556	$\bar{9}.74476n$
$w = 8$	$-$	5526	$\bar{9}.74241n$
$w = 16$	$-$	5560	$\bar{9}.74507n$
$\frac{d^3\alpha}{dt^3}$ for $w = 1$	$-$	5486	$\bar{9}.73926n$
$w = 8$	$-$	5551	$\bar{9}.74437n$
$w = 16$	$-$	5554	$\bar{9}.74461n$
$\frac{d^4\alpha}{dt^4}$ for $w = 1$	$-$	5551	$\bar{9}.74437n$
$w = 8$	$-$	5553	$\bar{9}.74453n$
$w = 16$	$-$	5554	$\bar{9}.74461n$

U^{VII}

		s.	
α for $w = 8$	$+$	0.0000000000001892	$\bar{7}.27692$
$w = 16$	$+$	2548	$\bar{7}.40620$
$\frac{d\alpha}{dt}$ for $w = 8$	$+$	1884	$\bar{7}.27509$
$w = 16$	$+$	3209	$\bar{7}.50637$
$\frac{d^2\alpha}{dt^2}$ for $w = 8$	$+$	3178	$\bar{7}.50215$
$w = 16$	$+$	3105	$\bar{7}.49206$
$\frac{d^3\alpha}{dt^3}$ for $w = 1$	$+$	2381	$\bar{7}.37676$
$w = 8$	$+$	3179	$\bar{7}.50229$
$w = 16$	$+$	3166	$\bar{7}.50051$
$\frac{d^4\alpha}{dt^4}$ for $w = 1$	$+$	3056	$\bar{7}.48515$
$w = 8$	$+$	3174	$\bar{7}.50161$
$w = 16$	$+$	3173	$\bar{7}.50147$

U^{VIII}

Initial Function.		s.	Logarithms.
α for $w = 16$	+	0.000000000000000677	$\bar{4}.83059$
$\frac{d\alpha}{dt}$ for $w = 8$	−	1040	$\bar{5}.01703n$
$w = 16$	+	464	$\bar{4}.66652$
$\frac{d^2\alpha}{dt^2}$ for $w = 8$	+	092	$\bar{5}.96379$
$w = 16$	+	207	$\bar{4}.31597$
$\frac{d^3\alpha}{dt^3}$ for $w = 8$	+	150	$\bar{4}.17609$
$w = 16$	+	184	$\bar{4}.26482$
$\frac{d^4\alpha}{dt^4}$ for $w = 1$	+	149	$\bar{4}.17319$
$w = 8$	+	158	$\bar{4}.19866$
$w = 16$	+	163	$\bar{4}.21219$

U^{IX}

		s.	
α for $w = 16$	−	0.00000000000000000277	$\bar{2}.44248n$
$\frac{d\alpha}{dt}$ for $w = 8$	−	2217	$\bar{3}.34576n$
$w = 16$	−	891	$\bar{2}.94988n$
$\frac{d^2\alpha}{dt^2}$ for $w = 8$	−	848	$\bar{2}.92840n$
$w = 16$	−	713	$\bar{2}.85309n$
$\frac{d^3\alpha}{dt^3}$ for $w = 8$	−	896	$\bar{2}.92531n$
$w = 16$	−	805	$\bar{2}.90580n$
$\frac{d^4\alpha}{dt^4}$ for $w = 8$	−	847	$\bar{2}.92788n$
$w = 16$	−	845	$\bar{2}.92686n$

U^{X}

		s.	
α for $w = 16$	−	0.0000000000000000000236	$\bar{0}.37291n$
$\frac{d^2\alpha}{dt^2}$ for $w = 8$	+	450	$\bar{0}.65321$
$w = 16$	+	156	$\bar{0}.19312$
$\frac{d^3\alpha}{dt^3}$ for $w = 8$	+	279	$\bar{0}.44560$
$w = 16$	+	192	$\bar{0}.28330$
$\frac{d^4\alpha}{dt^4}$ for $w = 8$	+	281	$\bar{0}.44871$
$w = 16$	+	259	$\bar{0}.41330$

U^{XI}

Initial Function.			Logarithms.
$\frac{d^3\alpha}{dt^3}$ for $w = 8$	+	s. 0.000000000000000000000272	$\overline{\overline{8}}.43457$
$w = 16$	+	63	$\overline{\overline{7}}.79934$
$\frac{d^4\alpha}{dt^4}$ for $w = 8$	+	125	$\overline{\overline{8}}.09691$
$w = 16$	+	119	$\overline{\overline{8}}.07555$

U^{XII}

$\frac{d^4\alpha}{dt^4}$ for $w = 8$	−	s. 0.00000000000000000000000112	$\overline{\overline{6}}.04922n$
$w = 16$	−	80	$\overline{\overline{5}}.90309n$

It will be advantageous to divide the computation of the terms $U^{I}t \ldots U^{n}t^{n}$ into two parts.

Representing the sum of the first five terms of the series by Y^4, we have

$$\alpha = \alpha_0 + U^{I}t + U^{II}t^2 + U^{III}t^3 + U^{IV}t^4 = Y_4,$$

in which the terms U^{I}, U^{II}, U^{III}, U^{IV}, are those marked with an asterisk on pp. 260, 261. Y_4 will now remain unchanged, whatever the results for the higher powers of the time.

We shall have, therefore,

$$\alpha = Y_4 + U^{V}t^5 + U^{VI}t^6 + U^{VII}t^7, \text{\&c.}$$

The computation of the terms $U^{I}t$, $U^{II}t^2$, $U^{III}t^3$, $U^{IV}t^4$, may be advantageously performed by the summation of the series of differences derived from the differential coefficients.

For the instant t_0 we have

$$\Delta_1 = \frac{d\alpha}{dt}, \quad \Delta_2 = 2\frac{d^2\alpha}{dt^2}, \quad \Delta_3 = 2.3\frac{d^3\alpha}{dt^3}, \quad \Delta_4 = 2.3.4\frac{d^4\alpha}{dt^4}, \quad \text{\&c.}$$

For the intervals w, $2w$, $3w$, $4w$, &c., we find:

		Δ_1	Δ_2
For $w = 1$,	$U^{\text{II}} t^2 = U^{\text{II}} t^2$		
		$U^{\text{II}}(2t+1)$	
$w = 2$,	$U^{\text{II}}(t+1)^2 = U^{\text{II}}(t^2 + 2t + 1)$		$2\,U^{\text{II}}$
		$U^{\text{II}}(2t+3)$	
$w = 3$,	$U^{\text{II}}(t+2)^2 = U^{\text{II}}(t^2 + 4t + 4)$		$2\,U^{\text{II}}$
		$U^{\text{II}}(2t+5)$	
$w = 4$,	$U^{\text{II}}(t+3)^2 = U^{\text{II}}(t^2 + 6t + 9)$		

The second differences are therefore constant, and for $w = 1$, $\Delta_2 = \dfrac{d^2\alpha}{dt^2}$. For any value of w we have $\Delta_2 = w^2\dfrac{d^2\alpha}{dt^2}$.

In like manner we shall find:

$$\Delta_3 = w^3\frac{d^3\alpha}{dt^3}, \qquad \Delta_4 = w^4\frac{d^4\alpha}{dt^4}, \qquad \Delta_5 = w^5\frac{d^5\alpha}{dt^5}, \qquad \text{\&c.}$$

For the terms following the fourth, it will be better to employ the logarithms of U^{V}, U^{VI}, &c., in obtaining the products $U^{\text{V}} t^5$, $U^{\text{VI}} t^6$, &c.

We shall always have the check that for these terms the differences Δ_5, Δ_6, &c., will be constant.

The values of $U^{\text{I}} t$ are found by adding the values of $\dfrac{d\alpha}{dt}$ successively to the initial function.

Having obtained the products $U^{\text{II}} t^2$ for three years, of $U^{\text{III}} t^3$ for four years, and of $U^{\text{IV}} t^4$ for five years, the succeeding terms will be obtained from the differences thereby derived.

The following example will serve as an illustration.

Given

$$U^{\text{II}} = -0\overset{s.}{.}1511894 \qquad U^{\text{III}} = -0\overset{s.}{.}00033958 \qquad \text{and} \qquad U^{\text{IV}} = +0\overset{s.}{.}0000034726$$

we have

	$U^{\text{II}} t^2$	Δ_1	Δ_2
	s.		
1876	−0.1511894		
		−0.4535682	
1877	−0.6047576		−0.3023788
		−0.7559470	
1878	−1.3607046		−0.3023788
		−1.0583258	
1879	−2.4190304		−0.3023788
		−1.3607046	
1880	−3.7797350		
	&c.		

	$U^{III}t^3$	Δ_1	Δ_2	Δ_3
	s.			
1876	−0.0003396			
		−0.0023770		
1877	−0.0027166		−0.0040750	
		−0.0064520		−0.0020375
1878	−0.0091686		−0.0061125	
		−0.0125645		−0.0020375
1879	−0.0217331		−0.0081500	
		−0.0207145		−0.0020375
1880	−0.0424476		−0.0101875	
		−0.0309020		
1881	−0.0733496			
	&c.			

	$U^{IV}t^4$	Δ_1	Δ_2	Δ_3	Δ_4
	s.				
1876	+0.0000035				
		+0.0000521			
1877	+0.0000556		+0.0001736		
		+0.0002257		+0.0002084	
1878	+0.0002813		+0.0003820		+0.0000833
		+0.0006077		+0.0002917	
1879	+0.0008890		+0.0006737		+0.0000833
		+0.0012814		+0.0003750	
1880	+0.0021704		+0.0010487		+0.0000833
		+0.0023301		+0.0004583	
1881	+0.0045005		+0.0015070		
		+0.0038371			
1882	+0.0083376				

The following values of Y_4 were obtained in the manner above indicated.

Column 7 contains the values of a derived from equations (6), designated by Y_0.

Column 8 contains the residuals $Y_0 - Y_4$.

REDUCTIONS FROM 1875 TO 1755.

$$1875.0 \quad \alpha_0 = 7^{h}\ 29^{m}\ 5^{s}.6310$$

Date.	$U^{I}t$	$U^{II}t^2$	$U^{III}t^3$	$U^{IV}t^4$	Y_4	Y_0	$Y_0 - Y_4$
	h. m. s.	m. s.	m. s.	m. s.	h. m. s.	m. s.	s.
1874	−0 1 13.5143	− 0 0.1512	+0 0.0003	+ 0 0.0000	7 27 51.966	. . . .	. . .
1873	−0 2 27.0286	− 0 0.6048	+0 0.0027	+ 0 0.0001	7 26 38.000	. . . .	. . .
1872	−0 3 40.5430	− 0 1.3607	+0 0.0092	+ 0 0.0003	7 25 23.737	. . . .	. . .
1871	−0 4 54.0573	− 0 2.4190	+0 0.0217	+ 0 0.0009	7 24 9.177	. . . .	. . .
1870	−0 6 7.5716	− 0 3.7797	+0 0.0424	+ 0 0.0022	7 22 54.324	. . . .	. . .
1869	−0 7 21.0859	− 0 5.4428	+0 0.0733	+ 0 0.0045	7 21 39.180	. . . .	. . .
1868	−0 8 34.6003	− 0 7.4083	+0 0.1165	+ 0 0.0083	7 20 23.747	. . . .	. . .
1867	−0 9 48.1146	− 0 9.6761	+0 0.1739	+ 0 0.0142	7 19 8.028	19 8.026	− 0.002
1866	−0 11 1.6289	− 0 12.2463	+0 0.2476	+ 0 0.0228	7 17 52.026	. . . .	. . .
1865	−0 12 15.1432	− 0 15.1189	+0 0.3396	+ 0 0.0347	7 16 35.743	. . . .	. . .
1864	−0 13 28.6576	− 0 18.2939	+0 0.4520	+ 0 0.0508	7 15 19.182	. . . .	. . .
1863	−0 14 42.1719	− 0 21.7713	+0 0.5868	+ 0 0.0720	7 14 2.347	. . . .	. . .
1862	−0 15 55.6862	− 0 25.5510	+0 0.7460	+ 0 0.0992	7 12 45.239	. . . .	. . .
1861	−0 17 9.2005	− 0 29.6331	+0 0.9318	+ 0 0.1334	7 11 27.862	. . . .	. . .
1860	−0 18 22.7149	− 0 34.0176	+0 1.1461	+ 0 0.1758	7 10 10.220	. . . .	. . .
1859	−0 19 36.2292	− 0 38.7045	+0 1.3909	+ 0 0.2276	7 8 52.316	8 52.315	− 0.001
1858	−0 20 49.7435	− 0 43.6937	+0 1.6683	+ 0 0.2900	7 7 34 152	. . . .	. . .
1857	−0 22 3.2578	− 0 48.9854	+0 1.9804	+ 0 0.3645	7 6 15.733	. . . .	. . .
1856	−0 23 16.7722	− 0 54.5794	+0 2.3292	+ 0 0.4526	7 4 57.061	. . . .	. . .
1855	−0 24 30.2865	− 1 0.4758	+0 2.7166	+ 0 0.5556	7 3 38.141	. . . .	. . .
1854	−0 25 43.8008	− 1 6.6745	+0 3.1448	+ 0 0.6754	7 2 18.976	. . . .	. . .
1853	−0 26 57.3151	− 1 13.1757	+0 3.6158	+ 0 0.8135	7 0 59.569	. . . .	. . .
1852	−0 28 10 8294	− 1 19.9792	+0 4.1316	+ 0 0.9718	6 59 39.926	. . . .	. . .
1851	−0 29 24.3438	− 1 27.0851	+0 4.6943	+ 0 1.1521	6 58 20.049	58 20.073	+ 0.024
1843	−0 39 12.4584	− 2 34.8180	+0 11.1273	+ 0 3.6413	6 47 33.123	47 33.225	+ 0.102
1835	−0 49 0 5730	− 4 1.9031	+0 21.7329	+ 0 8.8899	6 36 33.778	36 34.034	+ 0.256
1827	−0 58 48.6876	− 5 48.3404	+0 37.5545	+ 0 18.4340	6 25 24.591	25 25.071	+ 0 480
1819	−1 9 36.8022	− 7 54.1299	+0 59.6352	+ 0 34.1513	6 14 8.485	14 9.166	+ 0.681
1811	−1 18 24.9167	−10 19.2719	+1 29.0183	+ 0 58.2606	6 2 48.721	2 49.333	+ 0.612
1803	−1 28 13.0313	−13 3.7660	+2 6.7464	+ 1 33.3222	5 51 28.902	51 28.695	− 0.207
1795	−1 38 1.1459	−16 7.6123	+2 53.8634	+ 2 22.2377	5 40 12 974	40 10.397	− 2.577
1787	−1 47 49.2605	−19 30.8108	+3 51.4123	+ 3 28.2503	5 29 5.222	28 57.519	− 7.703
1779	−1 57 37.3751	−23 13.3616	+5 0.4359	+ 4 54.9440	5 18 10.274	17 52.998	−17.276
1771	−2 7 25.4897	−27 15.2649	+6 21.9782	+ 6 46.2451	5 7 33.100	6 59.542	−33.558
1763	−2 17 13 6043	−31 36.5203	+7 57.0812	+ 9 6.4202	4 57 19.008	56 19.577	−59.431
1755	−2 27 1.7189	−36 17.1276	+9 46.7889	+12 0.0782	4 47 33 652	45 55.203	−98.449

REDUCTIONS FROM 1875 TO 1955.

$$1875.0 \quad \alpha_0 = 7^{h}\ 29^{m}\ 5^{s}.6310$$

Date.	Y_4	Y_0	$Y_0 - Y_4$	Date.	Y_4	Y_0	$Y_0 - Y_4$
	h. m. s.	m. s.	s.		h. m. s.	m. s.	s.
1876	7 30 18.994	....	...	1892	7 49 10.302	....	...
1877	7 31 32.052	....	...	1893	7 50 18.288	....	...
1878	7 32 44.804	....	...	1894	7 51 25.947	....	...
1879	7 33 57.248	....	...	1895	7 52 33.281	....	...
1880	7 35 9.383	....	...	1896	7 53 40.288	....	...
1881	7 36 21.205	....	...	1897	7 54 46.968	....	...
1882	7 37 32.715	....	...	1898	7 55 53.321	....	...
1883	7 38 43.910	38 43.913	+ 0.003	1899	7 56 59.347	56 59.309	− 0.038
1884	7 39 54.789	....	...				
1885	7 41 5.350	....	...	1907	8 5 35.785	5 35.579	− 0.206
1886	7 42 15.593	....	...	1915	8 13 51.458	13 50.774	− 0.684
1887	7 43 25.517	....	...	1923	8 21 46.858	21 45.073	− 1.785
1888	7 44 35.119	....	...	1931	8 30 22.819	30 18.828	− 3.991
1889	7 45 44.400	....	...	1939	8 36 40.518	36 32.527	− 7.991
1890	7 46 55.358	....	...	1947	8 43 41.472	43 26.768	− 14.704
1891	7 48 1.992	48 1.994	+ 0.002	1955	8 50 27.539	50 2.231	− 25.308

Limitations in the Method of Development by Differential Coefficients.

That the method of development by means of differential coefficients expressed in terms of the ascending powers of the time has certain limitations in its application will be evident from an examination of the residuals $Y_0 - Y_4$ given on pages 267 and 268.

The extent of this limitation may be shown in the following manner. Let us assume that an error x occurs in the initial function where its effect will be a maximum. The magnitude of the errors in the successive orders of differences will be represented by the coefficients of x given in the following scheme: —

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}	Δ_{12}	
												$+x$	
											$+x$		
										$+x$		$-12x$	
									$+x$		$-11x$		
								$+x$		$-10x$		$+66x$	
							$+x$		$-9x$		$+55x$		
						$+x$		$-8x$		$+45x$		$-220x$	
					$+x$		$-7x$		$+36x$		$-165x$		
				$+x$		$-6x$		$+28x$		$-120x$		$+495x$	
			$+x$		$-5x$		$+21x$		$-84x$		$+330x$		
		$+x$		$-4x$		$+15x$		$-56x$		$+210x$		$-792x$	
	$+x$		$-3x$		$+10x$		$-35x$		$+126x$		$-462x$		
$F(\alpha+x)$		$-2x$		$+6x$		$-20x$		$+70x$		$-252x$		$+924x$	(51)
	$-x$		$+3x$		$-10x$		$+35x$		$-126x$		$+462x$		
		$+x$		$-4x$		$+15x$		$-56x$		$+210x$		$-792x$	
			$-x$		$+5x$		$-21x$		$+84x$		$-330x$		
				$+x$		$-6x$		$+28x$		$-120x$		$+495x$	
					$-x$		$+7x$		$-36x$		$+165x$		
						$+x$		$-8x$		$+45x$		$-220x$	
							$-x$		$+9x$		$-55x$		
								$+x$		$-10x$		$+66x$	
									$-x$		$+11x$		
										$+x$		$-12x$	
											$-x$		
												$+x$	

The total effect ϵ of the error x in changing the values of the differential coefficients will be expressed by the equation:

$$
\begin{aligned}
\varepsilon = & \frac{1}{2\,w^2}\left[-2x - \frac{1}{12}(+6x) + \frac{1}{90}(-20x) - \frac{1}{560}(+70x) + \frac{1}{3150}(-252x) - \ldots \&\text{c.}\right] \\
& \frac{1}{2.3.4\,w^4}\left[+6x - \frac{1}{6}(-20x) + \frac{7}{240}(+70x) - \frac{41}{7560}(-252x) + \ldots \&\text{c.}\right. \\
& \frac{1}{2.3.4.5.6\,w^6}\left[-20x - \frac{1}{4}(+70x) + \frac{13}{240}(-252x) - \ldots \&\text{c.}\right] \qquad (52) \\
& \frac{1}{2.3.4.5.6.7.8\,w^8}\left[+70x - \frac{1}{3}(-252x) + \ldots \&\text{c.}\right] \\
& \frac{1}{2.3.4.5.6.7.8.9.10\,w^{10}}\left[-252x - \ldots \&\text{c.}\right] \\
& \frac{1}{2.3.4.5.6.7.8.9.10.11.12\,w^{12}}\left[+924x + \ldots \&\text{c.}\right]
\end{aligned}
$$

Simplifying this equation, we have:

$$\varepsilon = -1.46361\frac{x}{w^2} + 0.53090\frac{x}{w^4} - 0.07105\frac{x}{w^6} + 0.0038194\frac{x}{w^8} - 0.00006944\frac{x}{w^{10}} + 0.0000019290\frac{x}{w^{12}} \tag{53}$$

$$= [0.16543n]\frac{x}{w^2} + [9.72501]\frac{x}{w^4} + [8.85156n]\frac{x}{w^6} + [7.58200]\frac{x}{w^8} + [5.84164n]\frac{x}{w^{10}} + [4.28534]\frac{x}{w^{12}}.$$

From equation (53) we derive the following numerical values of the effect of the error x for $w=1$, $w=8$, $w=16$, $w=30$, and $w=40$:

	$w=1$	$w=1$ Logarithms.	$w=8$	$w=8$ Logarithms.
U^{II}	$-1.4636x$	$[0.16543n]x$	$-0.022869x$	$[8.35925n]x$
U^{IV}	$+0.53090x$	$[9.72501]x$	$+0.00012961x$	$[6.11264]x$
U^{VI}	$-0.071049x$	$[8.85156n]x$	$-0.00000027103x$	$[3.43302n]x$
U^{VIII}	....	....	$+0.0000000002276x$	$[0.35729]x$
U^{X}	....	....	$-0.000000000000064676x$	$[\bar{6}.81074n]x$
U^{XII}	....	....	$+0.000000000000000028071x$	$[\bar{3}.44826]x$

	$w=16$	$w=16$ Logarithms.	$w=30$ Logarithms.	$w=40$ Logarithms.
U^{II}	$-0.0057173x$	$[7.75719n]x$	$[7.21119n]x$	$[6.96131n]x$
U^{IV}	$+0.0000081009x$	$[4.90853]x$	$[3.81653]x$	$[3.31677]x$
U^{VI}	$-0.0000000042349x$	$[1.62684n]x$	$[\bar{9}.98883n]x$	$[\bar{9}.23920n]x$
U^{VIII}	$+0.00000000000088929x$	$[\bar{7}.94904]x$	$[\bar{5}.76503]x$	$[\bar{4}.76552]x$
U^{X}	$-0.00000000000000000063160x$	$[\bar{3}.80044n]x$	$[\bar{\bar{1}}.07043n]x$	$[\bar{\bar{9}}.82104n]x$
U^{XII}	$+0.000000000000000000000068534x$	$[\bar{\bar{9}}.83591]x$	$[\bar{\bar{6}}.55989]x$	$[\bar{\bar{5}}.06062]x$

We illustrate the computation by assuming

$$x = \text{one unit in the third decimal place of } \alpha \text{ for 1875.0.}$$

Then, assuming

$$\begin{aligned} t &= 10 \text{ for } w = 1 \\ t &= 40 \text{ for } w = 8 \\ t &= 40 \text{ for } w = 16 \\ t &= 100 \text{ for } w = 16 \\ t &= 120 \text{ for } w = 16 \\ t &= 120 \text{ for } w = 30 \\ t &= 120 \text{ for } w = 40 \end{aligned}$$

we have for the total effect ϵ of the error x:

	$w=1$ $t=10$	$w=8$ $t=40$	$w=16$ $t=40$	$w=16$ $t=100$	$w=16$ $t=120$	$w=30$ $t=120$	$w=40$ $t=120$
	s.	s.	s.	s.	s.	s.	s.
$U^{II}t^2$	− 0.146	−0.037	−0.009	−0.057	− 0.082	−0.023	−0.013
$U^{IV}t^4$	+ 5.309	+0.332	+0.021	+0.810	+ 1.680	+0.136	+0.043
$U^{VI}t^6$	−71.049	−1.110	−0.017	−4.235	−12.645	−0.291	−0.052
$U^{VIII}t^8$	. . . .	+1.492	+0.006	+8.893	+38.238	+0.250	+0.025
$U^{X}t^{10}$	. . . .	−0.678	−0.001	−6.316	−39.107	−0.073	−0.004
$U^{XII}t^{12}$	. . . .	+0.471	+0.000	+6.853	+61.106	+0.032	+0.001
Sums		+0.470	+0.000	+5.948	+49.190	+0.031	+0.000

The decided advantage in the choice of a large value for the interval w is very obvious from these results. It is, however, important to be remembered that these values of ϵ are the result of a single error producing a maximum effect. The errors which occur ordinarily in the use of logarithmic tables are likely to be distributed with considerable regularity, and these errors will probably neutralize each other to some extent.

TABULAR VALUES OF THE LOGARITHMS OF THE FIFTH AND HIGHER POWERS OF THE TIME FROM 8 TO 120 YEARS.

t	log t^5	log t^6	log t^7	log t^8	log t^9	log t^{10}	log t^{11}	log t^{12}
8	4.51545	5.41854	6.32163	7.22472	8.12781	9.03090	9.93399	10.83708
9	4.77121	5.72545	6.67970	7.63394	8.58818	9.54242	10.49667	11.45091
10	5.00000	6.00000	7.00000	8.00000	9.00000	10.00000	11.00000	12.00000
11	5.20696	6.24836	7.28975	8.33114	9.37253	10.41393	11.45532	12.49671
12	5.39591	6.47509	7.55427	8.63345	9.71263	10.79182	11.87099	12.95017
13	5.56972	6.68366	7.79760	8.91155	10.02549	11.13943	12.25338	13.36732
14	5.73064	6.87677	8.02290	9.16902	10.31515	11.46128	12.60741	13.75354
15	5.88046	7.05655	8.23264	9.40873	10.58482	11.76091	12.93700	14.11310
16	6.02060	7.22472	8.42884	9.63296	10.83708	12.04120	13.24532	14.44944
17	6.15224	7.38269	8.61314	9.84359	11.07404	12.30449	13.53494	14.76539
18	6.27636	7.53164	8.78691	10.04218	11.29745	12.55272	13.80800	15.06327
19	6.39377	7.67252	8.95128	10.23003	11.50878	12.78754	14.06629	15.34504
20	6.50515	7.80618	9.10721	10.40824	11.70927	13.01030	14.31133	15.61236
21	6.61110	7.93332	9.25553	10.57775	11.89997	13.22219	14.54441	15.86663
22	6.71211	8.05454	9.39696	10.73938	12.08180	13.42423	14.76665	16.10907
23	6.80864	8.17037	9.53209	10.89382	12.25555	13.61728	14.97901	16.34073
24	6.90106	8.28127	9.66148	11.04169	12.42190	13.80211	15.18232	16.56253
32	7.52575	9.03090	10.53605	12.04120	13.54635	15.05150	16.55665	18.06180
40	8.01030	9.61236	11.21442	12.81648	14.41854	16.02060	17.62266	19.22472
48	8.40621	10.08745	11.76869	13.44993	15.13117	16.81241	18.49365	20.17489
56	8.74094	10.48913	12.23732	13.98550	15.73369	17.48188	19.23007	20.97826
64	9.03090	10.83708	12.64326	14.44944	16.25562	18.06180	19.86798	21.67416
72	9.28666	11.14399	13.00133	14.85866	16.71599	18.57332	20.43066	22.28799
80	9.51545	11.41854	13.32163	15.22472	17.12781	19.03090	20.93399	22.83708
88	9.72241	11.66690	13.61138	15.55586	17.50034	19.44483	21.38931	23.33379
96	9.91136	11.89363	13.87590	15.85817	17.84044	19.82271	21.80498	23.78725
104	10.08517	12.10220	14.11923	16.13627	18.15330	20.17033	22.18737	24.20440
112	10.24609	12.29531	14.34453	16.39374	18.44296	20.49218	22.54140	24.59062
120	10.39591	12.47509	14.55427	16.63345	18.71263	20.79181	22.87099	24.95017

By the aid of this table the following values of $U^{V} t^5 \ldots . U^{XII} t^{12}$ were computed with the values of $U^{V} \ldots . U^{XII}$ given on pp. 261–264.

Representing the initial function by J we have:

$U^{V} t^5$.

Epoch.		$J = a$	$J = \frac{da}{dt}$	$J = \frac{d^2 a}{dt^2}$	$J = \frac{d^3 a}{dt^3}$	$J = \frac{d^4 a}{dt^4}$
		s.	s.	s.	s.	s.
1851	$w =$ 1	....	....	+ 0.041	+ 0.041	+ 0.041
	8	+ 0.047	+ 0.042	+ 0.040	+ 0.041	+ 0.041
	16	+ 0.039	+ 0.041	+ 0.041	+ 0.041	+ 0.041
1843	$w =$ 1	....	....	+ 0.170	+ 0.174	+ 0.174
	8	+ 0.198	+ 0.177	+ 0.174	+ 0.174	+ 0.173
	16	+ 0.166	+ 0.174	+ 0.173	+ 0.174	+ 0.173
1835	$w =$ 1	....	....	+ 0.521	+ 0.530	+ 0.530
	8	+ 0.603	+ 0.541	+ 0.530	+ 0.530	+ 0.529
	16	+ 0.505	+ 0.531	+ 0.529	+ 0.530	+ 0.529
1827	$w =$ 1	....	....	+ 1.295	+ 1.319	+ 1.318
	8	+ 1.502	+ 1.346	+ 1.320	+ 1.320	+ 1.317
	16	+ 1.258	+ 1.321	+ 1.317	+ 1.320	+ 1.317
1819	$w =$ 1	....	....	+ 2.799	+ 2.851	+ 2.848
	8	+ 3.245	+ 2.910	+ 2.852	+ 2.853	+ 2.847
	16	+ 2.718	+ 2.855	+ 2.847	+ 2.853	+ 2.847
1811	$w =$ 1	....	....	+ 5.458	+ 5.559	+ 5.553
	8	+ 6.328	+ 5.674	+ 5.561	+ 5.563	+ 5.550
	16	+ 5.300	+ 5.566	+ 5.550	+ 5.562	+ 5.550
1803	$w =$ 1	....	....	+ 9.835	+ 10.017	+ 10.007
	8	+ 11.403	+ 10.224	+ 10.021	+ 10.025	+ 10.002
	16	+ 9.551	+ 10.030	+ 10.002	+ 10.023	+ 10.002
1795	$w =$ 1	....	....	+ 16.656	+ 16.964	+ 16.948
	8	+ 19.310	+ 17.315	+ 16.970	+ 16.977	+ 16.938
	16	+ 16.174	+ 16.987	+ 16.938	+ 16.974	+ 16.938
1787	$w =$ 1	....	....	+ 26.824	+ 27.321	+ 27.294
	8	+ 31.099	+ 27.885	+ 27.331	+ 27.341	+ 27.278
	16	+ 26.049	+ 27.357	+ 27.278	+ 27.336	+ 27.278
1779	$w =$ 1	....	....	+ 41.446	+ 42.212	+ 42.172
	8	+ 48.051	+ 43.085	+ 42.228	+ 42.244	+ 42.147
	16	+ 40.248	+ 42.269	+ 42.147	+ 42.237	+ 42.147
1771	$w =$ 1	....	....	+ 61.843	+ 62.987	+ 62.926
	8	+ 71.699	+ 64.288	+ 63.011	+ 63.035	+ 62.890
	16	+ 60.056	+ 63.071	+ 62.890	+ 63.024	+ 62.890
1763	$w =$ 1	....	....	+ 89.580	+ 91.237	+ 91.148
	8	+ 103.858	+ 93.122	+ 91.271	+ 91.307	+ 91.097
	16	+ 86.990	+ 91.359	+ 91.097	+ 91.289	+ 91.097
1755	$w =$ 1	....	....	+ 126.483	+ 128.820	+ 128.698
	8	+ 146.639	+ 131.485	+ 128.870	+ 128.920	+ 128.623
	16	+ 122.826	+ 128.994	+ 128.623	+ 128.896	+ 128.623

$U^{VI} t^6$.

Epoch.		$J = a$	$J = \frac{d a}{d t}$	$J = \frac{d^2 a}{d t^2}$	$J = \frac{d^3 a}{d t^3}$	$J = \frac{d^4 a}{d t^4}$
		s.	s.	s.	s.	s.
1851	w = 1	....	....	− 0.011	− 0.010	− 0.011
	8	− 0.007	− 0.010	− 0.011	− 0.011	− 0.011
	16	− 0.011	− 0.011	− 0.011	− 0.011	− 0.011
1843	w = 1	....	....	− 0.060	− 0.059	− 0.060
	8	− 0.040	− 0.057	− 0.059	− 0.060	− 0.060
	16	− 0.061	− 0.061	− 0.060	− 0.060	− 0.060
1835	w = 1	....	....	− 0.228	− 0.225	− 0.227
	8	− 0.152	− 0.216	− 0.226	− 0.227	− 0.227
	16	− 0.234	− 0.232	− 0.228	− 0.228	− 0.228
1827	w = 1	....	....	− 0.680	− 0.671	− 0.679
	8	− 0.454	− 0.646	− 0.676	− 0.679	− 0.679
	16	− 0.700	− 0.691	− 0.680	− 0.682	− 0.682
1819	w = 1	....	....	− 1.714	− 1.692	− 1.712
	8	− 1.144	− 1.629	− 1.704	− 1.712	− 1.713
	16	− 1.766	− 1.743	− 1.715	− 1.713	− 1.713
1811	w = 1	....	....	− 3.818	− 3.770	− 3.815
	8	− 2.549	− 3.629	− 3.797	− 3.815	− 3.816
	16	− 3.934	− 3.884	− 3.821	− 3.817	− 3.817
1803	w = 1	....	....	− 7.740	− 7.643	− 7.733
	8	− 5.167	− 7.357	− 7.698	− 7.733	− 7.736
	16	− 7.976	− 7.874	− 7.746	− 7.738	− 7.738
1795	w = 1	....	....	− 14.565	− 14.381	− 14.552
	8	− 9.723	− 13.844	− 14.486	− 14.552	− 14.557
	16	− 15.008	− 14.816	− 14.575	− 14.560	− 14.560
1787	w = 1	....	....	− 25.802	− 25.478	− 25.779
	8	− 17.225	− 24.526	− 25.663	− 27.779	− 25.789
	16	− 26.588	− 26.248	− 25.821	− 25.794	− 25.794
1779	w = 1	....	....	− 43.490	− 42.943	− 43.451
	8	− 29.033	− 41.338	− 43.255	− 43.451	− 43.467
	16	− 44.814	− 44.242	− 43.521	− 43.475	− 43.475
1771	w = 1	....	....	− 70.301	− 69.416	− 70.238
	8	− 46.931	− 66.822	− 69.922	− 70.238	− 70.265
	16	− 72.441	− 71.516	− 70.351	− 70.277	− 70.277
1763	w = 1	....	....	−109.667	−108.285	−109.567
	8	− 73.210	−104.240	−109.076	−109.567	−109.607
	16	−113.002	−111.560	−109.746	−109.628	−109.628
1755	w = 1	....	....	−165.900	−163.814	−165.753
	8	−110.752	−157.693	−165.007	−165.753	−165.814
	16	−170.950	−168.768	−166.020	−165.844	−165.844

$U^{VII} t^7$.

Epoch.		$J = a$	$J = \frac{d a}{d t}$	$J = \frac{d^2 a}{d t^2}$	$J = \frac{d^3 a}{d t^3}$	$J = \frac{d^4 a}{d t^4}$
		s.	s.	s.	s.	s.
1851	w = 1	. . . .	. . . .	. . . .	− 0.001	− 0.001
	8	− 0.001	− 0.001	− 0.001	− 0.001	− 0.001
	16	− 0.001	− 0.001	− 0.001	− 0.001	− 0.001
1843	w = 1	. . . .	. . . .	. . . .	− 0.008	− 0.010
	8	− 0.006	− 0.006	− 0.011	− 0.011	− 0.011
	16	− 0.009	− 0.011	− 0.011	− 0.011	− 0.011
1835	w = 1	. . . .	. . . .	. . . .	− 0.039	− 0.050
	8	− 0.031	− 0.031	− 0.052	− 0.052	− 0.052
	16	− 0.042	− 0.053	− 0.051	− 0.052	− 0.052
1827	w = 1	. . . .	. . . .	. . . .	− 0.140	− 0.179
	8	− 0.111	− 0.111	− 0.187	− 0.187	− 0.186
	16	− 0.150	− 0.188	− 0.182	− 0.186	− 0.186
1819	w = 1	. . . .	. . . .	. . . .	− 0.411	− 0.528
	8	− 0.327	− 0.325	− 0.549	− 0.549	− 0.548
	16	− 0.440	− 0.554	− 0.536	− 0.547	− 0.548
1811	w = 1	. . . .	. . . .	. . . .	− 1.047	− 1.344
	8	− 0.832	− 0.829	− 1.398	− 1.398	− 1.396
	16	− 1.121	− 1.411	− 1.366	− 1.392	− 1.396
1803	w = 1	. . . .	. . . .	. . . .	− 2.388	− 3.065
	8	− 1.898	− 1.890	− 3.188	− 3.189	− 3.184
	16	− 2.556	− 3.219	− 3.114	− 3.176	− 3.183
1795	w = 1	. . . .	. . . .	. . . .	− 4.993	− 6.409
	8	− 3.968	− 3.951	− 6.665	− 6.667	− 6.656
	16	− 5.343	− 6.730	− 6.512	− 6.639	− 6.654
1787	w = 1	. . . .	. . . .	. . . .	− 9.731	− 12.489
	8	− 7.732	− 7.700	− 12.988	− 12.992	− 12.972
	16	− 10.413	− 13.114	− 12.689	− 12.939	− 12.967
1779	w = 1	. . . .	. . . .	. . . .	− 17.892	− 22.964
	8	− 14.217	− 14.158	− 23.881	− 23.889	− 23.851
	16	− 19.147	− 24.114	− 23.333	− 23.791	− 23.844
1771	w = 1	. . . .	. . . .	. . . .	− 31.332	− 40.214
	8	− 24.897	− 24.793	− 41.820	− 41.833	− 41.768
	16	− 33.530	− 42.228	− 40.860	− 41.662	− 41.754
1763	w = 1	. . . .	. . . .	. . . .	− 52.637	− 67.558
	8	− 41.827	− 41.651	− 70.256	− 70.278	− 70.168
	16	− 56.329	− 70.942	− 68.642	− 69.991	− 70.146
1755	w = 1	. . . .	. . . .	. . . .	− 85.316	−109.502
	8	− 67.794	− 67.508	−113.873	−113.909	−113.732
	16	− 91.300	−114.984	−111.258	−113.444	−113.695

$U^{\text{VIII}}\, t^8$.

Epoch.		$J = a$	$J = \frac{d\,a}{d\,t}$	$J = \frac{d^2\,a}{d\,t^2}$	$J = \frac{d^3\,a}{d\,t^3}$	$J = \frac{d^4\,a}{d\,t^4}$
		s.	s.	s.	s.	s.
1851	$w =$ 1	....	....	....	....	+ 0.000
	8	....	− 0.000	+ 0.000	+ 0.000	+ 0.000
	16	+ 0.000	+ 0.000	+ 0.000	+ 0.000	+ 0.000
1843	$w =$ 1	....	....	....	....	+ 0.000
	8	....	− 0.001	+ 0.000	+ 0.000	+ 0.000
	16	+ 0.001	+ 0.000	+ 0.000	+ 0.000	+ 0.000
1835	$w =$ 1	....	....	....	....	+ 0.001
	8	....	− 0.007	+ 0.001	+ 0.001	+ 0.001
	16	+ 0.004	+ 0.003	+ 0.001	+ 0.001	+ 0.001
1827	$w =$ 1	....	....	....	....	+ 0.004
	8	....	− 0.029	+ 0.003	+ 0.004	+ 0.004
	16	+ 0.019	+ 0.013	+ 0.006	+ 0.005	+ 0.005
1819	$w =$ 1	....	....	....	....	+ 0.014
	8	....	− 0.101	+ 0.009	+ 0.014	+ 0.015
	16	+ 0.065	+ 0.045	+ 0.020	+ 0.018	+ 0.016
1811	$w =$ 1	....	....	....	....	+ 0.042
	8	....	− 0.293	+ 0.026	+ 0.042	+ 0.044
	16	+ 0.191	+ 0.131	+ 0.058	+ 0.052	+ 0.046
1803	$w =$ 1	....	....	....	....	+ 0.108
	8	....	− 0.751	+ 0.066	+ 0.108	+ 0.114
	16	+ 0.489	+ 0.335	+ 0.150	+ 0.133	+ 0.118
1795	$w =$ 1	....	....	....	....	+ 0.250
	8	....	− 1.744	+ 0.154	+ 0.252	+ 0.265
	16	+ 1.136	+ 0.778	+ 0.347	+ 0.309	+ 0.274
1787	$w =$ 1	....	....	....	....	+ 0.536
	8	....	− 3.740	+ 0.331	+ 0.539	+ 0.568
	16	+ 2.435	+ 1.669	+ 0.744	+ 0.662	+ 0.586
1779	$w =$ 1	....	....	....	....	+ 1.075
	8	....	− 7.502	+ 0.664	+ 1.082	+ 1.140
	16	+ 4.884	+ 3.347	+ 1.493	+ 1.327	+ 1.176
1771	$w =$ 1	....	....	....	....	+ 2.039
	8	....	− 14.233	+ 1.259	+ 2.053	+ 2.162
	16	+ 9.265	+ 6.350	+ 2.833	+ 2.518	+ 2.231
1763	$w =$ 1	....	....	....	....	+ 3.689
	8	....	− 25.750	+ 2.278	+ 3.714	+ 3.912
	16	+ 16.762	+ 11.488	+ 5.125	+ 4.556	+ 4.036
1755	$w =$ 1	....	....	....	....	+ 6.407
	8	....	− 44.718	+ 3.956	+ 6.450	+ 6.794
	16	+ 29.110	+ 19.951	+ 8.901	+ 7.912	+ 7.009

$U^{IX} t^9$.

Epoch.		$J = \alpha$	$J = \frac{d\alpha}{dt}$	$J = \frac{d^2\alpha}{dt^2}$	$J = \frac{d^3\alpha}{dt^3}$	$J = \frac{d^4\alpha}{dt^4}$
		s.	s.	s.	s.	s.
1851	$w =$ 8	....	+ 0.000	+ 0.000	+ 0.000	+ 0.000
	16	+ 0.000	+ 0.000	+ 0.000	+ 0.000	+ 0.000
1843	$w =$ 8	....	+ 0.001	+ 0.000	+ 0.000	+ 0.000
	16	+ 0.000	+ 0.000	+ 0.000	+ 0.000	+ 0.000
1835	$w =$ 8	....	+ 0.006	+ 0.002	+ 0.002	+ 0.002
	16	+ 0.001	+ 0.002	+ 0.002	+ 0.002	+ 0.002
1827	$w =$ 8	....	+ 0.030	+ 0.012	+ 0.011	+ 0.011
	16	+ 0.004	+ 0.012	+ 0.010	+ 0.011	+ 0.011
1819	$w =$ 8	....	+ 0.120	+ 0.046	+ 0.046	+ 0.048
	16	+ 0.015	+ 0.048	+ 0.039	+ 0.044	+ 0.046
1811	$w =$ 8	....	+ 0.399	+ 0.153	+ 0.152	+ 0.153
	16	+ 0.050	+ 0.160	+ 0.128	+ 0.145	+ 0.152
1803	$w =$ 8	....	+ 1.153	+ 0.440	+ 0.438	+ 0.440
	16	+ 0.144	+ 0.463	+ 0.371	+ 0.419	+ 0.439
1795	$w =$ 8	....	+ 2.976	+ 1.138	+ 1.130	+ 1.137
	16	+ 0.372	+ 1.196	+ 0.957	+ 1.080	+ 1.134
1787	$w =$ 8	....	+ 7.016	+ 2.684	+ 2.665	+ 2.680
	16	+ 0.877	+ 2.820	+ 2.257	+ 2.548	+ 2.674
1779	$w =$ 8	....	+15.354	+ 5.873	+ 5.831	+ 5.866
	16	+ 1.918	+ 6.170	+ 4.938	+ 5.575	+ 5.852
1771	$w =$ 8	....	+31.555	+12.070	+11.984	+12.055
	16	+ 3.943	+12.682	+10.148	+11.457	+12.027
1763	$w =$ 8	....	+61.479	+23.516	+23.349	+23.488
	16	+ 7.681	+24.708	+19.772	+22.323	+23.433
1755	$w =$ 8	....	+114.391	+43.755	+43.445	+43.703
	16	+14.293	+45.974	+36.789	+41.537	+43.600

$U^{X}\ t^{10}$.

Epoch.		$J=\alpha$	$J=\frac{d^2\alpha}{dt^2}$	$J=\frac{d^3\alpha}{dt^3}$	$J=\frac{d^4\alpha}{dt^4}$
		s.	s.	s.	s.
1851	$w=$ 8	....	+ 0.000	+ 0.000	+ 0.000
	16	− 0.000	+ 0.000	+ 0.000	+ 0.000
1843	$w=$ 8	....	+ 0.000	+ 0.000	+ 0.000
	16	− 0.000	+ 0.000	+ 0.000	+ 0.000
1835	$w=$ 8	....	+ 0.000	+ 0.000	+ 0.000
	16	− 0.000	+ 0.000	+ 0.000	+ 0.000
1827	$w=$ 8	....	+ 0.003	+ 0.002	+ 0.002
	16	− 0.002	+ 0.001	+ 0.001	+ 0.002
1819	$w=$ 8	....	+ 0.014	+ 0.008	+ 0.008
	16	− 0.007	+ 0.005	+ 0.006	+ 0.008
1811	$w=$ 8	....	+ 0.052	+ 0.032	+ 0.032
	16	− 0.027	+ 0.018	+ 0.022	+ 0.030
1803	$w=$ 8	....	+ 0.168	+ 0.104	+ 0.105
	16	− 0.088	+ 0.058	+ 0.072	+ 0.097
1795	$w=$ 8	....	+ 0.483	+ 0.300	+ 0.302
	16	− 0.253	+ 0.168	+ 0.206	+ 0.278
1787	$w=$ 8	....	+ 1.253	+ 0.777	+ 0.783
	16	− 0.657	+ 0.434	+ 0.535	+ 0.721
1779	$w=$ 8	....	+ 2.992	+ 1.855	+ 1.868
	16	− 1.569	+ 1.037	+ 1.276	+ 1.722
1771	$w=$ 8	....	+ 6.661	+ 4.130	+ 4.159
	16	− 3.493	+ 2.309	+ 2.842	+ 3.834
1763	$w=$ 8	....	+13.976	+ 8.665	+ 8.727
	16	− 7.330	+ 4.845	+ 5.963	+ 8.044
1755	$w=$ 8	....	+27.863	+17.275	+17.399
	16	−14.613	+ 9.659	+11.888	+16.037

Epoch.		$U^{\mathrm{XI}}\,t^{11}$. $J=\frac{d^3\alpha}{d\,t^3}$	$J=\frac{d^4\alpha}{d\,t^4}$	$U^{\mathrm{XII}}\,t^{12}$. $J=\frac{d^4\alpha}{d\,t^4}$
		s.	s.	s.
1851	$w=$ 8	− 0.000	− 0.000	− 0.000
	16	− 0.000	− 0.000	− 0.000
1843	$w=$ 8	− 0.000	− 0.000	− 0.000
	16	− 0.000	− 0.000	− 0.000
1835	$w=$ 8	− 0.000	− 0.000	− 0.000
	16	− 0.000	− 0.000	− 0.000
1827	$w=$ 8	− 0.001	− 0.000	− 0.000
	16	− 0.000	− 0.000	− 0.000
1819	$w=$ 8	− 0.005	− 0.002	− 0.001
	16	− 0.001	− 0.002	− 0.001
1811	$w=$ 8	− 0.020	− 0.009	− 0.005
	16	− 0.005	− 0.009	− 0.004
1803	$w=$ 8	− 0.073	− 0.034	− 0.022
	16	− 0.017	− 0.032	− 0.016
1795	$w=$ 8	− 0.234	− 0.107	− 0.077
	16	− 0.054	− 0.102	− 0.055
1787	$w=$ 8	− 0.667	− 0.306	− 0.242
	16	− 0.154	− 0.292	− 0.172
1779	$w=$ 8	− 1.736	− 0.798	− 0.686
	16	− 0.402	− 0.760	− 0.490
1771	$w=$ 8	− 4.187	− 1.924	− 1.793
	16	− 0.970	− 1.832	− 1.281
1763	$w=$ 8	− 9.462	− 4.348	− 4.364
	16	− 2.192	− 4.140	− 3.117
1755	$w=$ 8	− 20.210	− 9.287	− 9.986
	16	− 4.681	− 8.842	− 7.133

Sum of the Terms $U^{V} t^5 + U^{VI} t^6 + U^{VII} t^7 + \&c. \ldots U^{XII} t^{12}$.

Designating the sum of the terms, from the fifth forward, by the number of the last term, we have:

Epoch.	$t - 1875$.		$J = \alpha$	$J = \frac{d\alpha}{dt}$	$J = \frac{d^2\alpha}{dt^2}$	$J = \frac{d^3\alpha}{dt^3}$	$J = \frac{d^4\alpha}{dt^4}$
			s.	s.	s.	s.	s.
1851	− 24	$w =$ 1	. . . .	. . . .	VI + 0.030	VII + 0.030	VIII + 0.029
		8	VII + 0.039	IX + 0.031	X + 0.028	XI + 0.029	XII + 0.029
		16	X + 0.027	IX + 0.029	X + 0.029	XI + 0.029	XII + 0.029
1843	− 32	$w =$ 1	. . . .	. . . .	VI + 0.111	VII + 0.107	VIII + 0.104
		8	VII + 0.152	IX + 0.114	X + 0.104	XI + 0.103	XII + 0.102
		16	X + 0.097	IX + 0.102	X + 0.102	XI + 0.103	XII + 0.102
1835	− 40	$w =$ 1	. . . .	. . . .	VI + 0.293	VII + 0.266	VIII + 0.254
		8	VII + 0.420	IX + 0.293	X + 0.255	XI + 0.254	XII + 0.253
		16	X + 0.234	IX + 0.251	X + 0.253	XI + 0.253	XII + 0.252
1827	− 48	$w =$ 1	. . . .	. . . .	VI + 0.615	VII + 0.508	VIII + 0.464
		8	VII + 0.937	IX + 0.590	X + 0.475	XI + 0.470	XII + 0.469
		16	X + 0.429	IX + 0.467	X + 0.472	XI + 0.469	XII + 0.467
1819	− 56	$w =$ 1	. . . .	. . . .	VI + 1.085	VII + 0.748	VIII + 0.622
		8	VII + 1.774	IX + 0.975	X + 0.668	XI + 0.655	XII + 0.654
		16	X + 0.585	IX + 0.651	X + 0.660	XI + 0.660	XII + 0.653
1811	− 64	$w =$ 1	. . . .	. . . .	VI + 1.640	VII + 0.742	VIII + 0.436
		8	VII + 2.947	IX + 1.322	X + 0.597	XI + 0.556	XII + 0.553
		16	X + 0.459	IX + 0.562	X + 0.567	XI + 0.567	XII + 0.552
1803	− 72	$w =$ 1	. . . .	. . . .	VI + 2.095	VII − 0.014	VIII − 0.683
		8	VII + 4.338	IX + 1.379	X − 0.191	XI − 0.320	XII − 0.315
		16	X − 0.436	IX − 0.265	X − 0.279	XI − 0.284	XII − 0.313
1795	− 80	$w =$ 1	. . . .	. . . .	VI + 2.091	VII − 2.410	VIII − 3.763
		8	VII + 5.619	IX + 0.752	X − 2.406	XI − 2.794	XII − 2.755
		16	X − 2.922	IX − 2.585	X − 2.677	XI − 2.684	XII − 2.747
1787	− 88	$w =$ 1	. . . .	. . . .	VI + 1.022	VII − 7.888	VIII − 10.438
		8	VII + 6.142	IX − 1.065	X − 7.052	XI − 8.116	XII − 8.000
		16	X − 8.297	IX − 7.516	X − 7.797	XI − 7.806	XII − 7.966
1779	− 96	$w =$ 1	. . . .	. . . .	VI − 2.044	VII − 18.623	VIII − 23.168
		8	VII + 4.801	IX − 4.559	X − 15.379	XI − 18.064	XII − 17.781
		16	X − 18.480	IX − 16.570	X − 17.239	XI − 17.253	XII − 17.672
1771	− 104	$w =$ 1	. . . .	. . . .	VI − 8.458	VII − 37.761	VIII − 45.487
		8	VII − 0.129	IX − 10.005	X − 28.741	XI − 35.056	XII − 34.484
		16	X − 36.200	IX − 31.641	X − 33.031	XI − 33.068	XII − 34.162
1763	− 112	$w =$ 1	. . . .	. . . .	VI − 20.087	VII − 69.685	VIII − 82.288
		8	VII − 11.179	IX − 17.040	X − 48.291	XI − 62.272	XII − 61.263
		16	X − 65.228	IX − 54.947	X − 57.549	XI − 57.680	XII − 60.421
1755	− 120	$w =$ 1	. . . .	. . . .	VI − 39.417	VII − 120.310	VIII − 140.150
		8	VII − 31.907	IX − 24.043	X − 74.436	XI − 103.782	XII − 102.300
		16	X − 110.634	IX − 88.833	X − 93.306	XI − 93.736	XII − 100.245

Sums of Y_4 and succeeding Terms.

Designating the sums of the terms of the entire series by the number of the last term, we have:

Epoch.		h. m. s.		$J = \alpha$	$J = \frac{d\alpha}{dt}$	$J = \frac{d^2\alpha}{dt^2}$	$J = \frac{d^3\alpha}{dt^3}$	$J = \frac{d^4\alpha}{dt^4}$
				m. s.	m. s.	m. s.	m. s.	m. s.
1851	Y_4	6 58 20.048	$w = 1$	... .	... .	Y_6 58 20.078	Y_7 58 20.078	Y_8 58 20.077
	Y_0	6 58 20.073	8	Y_7 58 20.087	Y_9 58 20.079	Y_{10} 58 20.076	Y_{11} 58 20 077	Y_{12} 58 20.077
			16	Y_{10} 58 20.075	Y_9 58 20.077	Y_{10} 58 20.077	Y_{11} 58 20.077	Y_{12} 58 20.077
1843	Y_4	6 47 33.123	$w = 1$	... .	... .	Y_6 47 33.234	Y_7 47 33.230	Y_8 47 33.227
	Y_0	6 47 33.225	8	Y_7 47 33.275	Y_9 47 33.237	Y_{10} 47 33.227	Y_{11} 47 33.226	Y_{12} 47 33.225
			16	Y_{10} 47 33.220	Y_9 47 33.225	Y_{10} 47 33.225	Y_{11} 47 33.226	Y_{12} 47 33.225
1835	Y_4	6 36 33.778	$w = 1$	... .	... .	Y_6 36 34.071	Y_7 36 34.044	Y_8 36 34.032
	Y_0	6 36 34.034	8	Y_7 36 34.198	Y_9 36 34.071	Y_{10} 36 34.033	Y_{11} 36 34.032	Y_{12} 36 34.031
			16	Y_{10} 36 34.012	Y_9 36 34.029	Y_{10} 36 34.031	Y_{11} 36 34.031	Y_{12} 36 34.030
1827	Y_4	6 25 24.592	$w = 1$	... .	... .	Y_6 25 25.207	Y_7 25 25.100	Y_8 25 25.056
	Y_0	6 25 25.071	8	Y_7 25 25.529	Y_9 25 25.182	Y_{10} 25 25.067	Y_{11} 25 25.062	Y_{12} 25 25.061
			16	Y_{10} 25 25.021	Y_9 25 25.059	Y_{10} 25 25.064	Y_{11} 25 25.061	Y_{12} 25 25 059
1819	Y_4	6 14 8.485	$w = 1$	... .	... .	Y_6 14 9.570	Y_7 14 9.233	Y_8 14 9.107
	Y_0	6 14 9.166	8	Y_7 14 10.259	Y_9 14 9.460	Y_{10} 14 9.153	Y_{11} 14 9.140	Y_{12} 14 9.139
			16	Y_{10} 14 9.070	Y_9 14 9.136	Y_{10} 14 9.145	Y_{11} 14 9.145	Y_{12} 14 9.138
1811	Y_4	6 2 48.721	$w = 1$	... .	... .	Y_6 2 50.361	Y_7 2 49.463	Y_8 2 49.157
	Y_0	6 2 49.333	8	Y_7 2 51.668	Y_9 2 50.043	Y_{10} 2 49.318	Y_{11} 2 49.277	Y_{12} 2 49.274
			16	Y_{10} 2 49.180	Y_9 2 49.283	Y_{10} 2 49.288	Y_{11} 2 49.288	Y_{12} 2 49.273
1803	Y_4	5 51 28.902	$w = 1$	... .	... .	Y_6 51 30.997	Y_7 51 28.888	Y_8 51 28.219
	Y_0	5 51 28.695	8	Y_7 51 33.240	Y_9 51 30.281	Y_{10} 51 28.711	Y_{11} 51 28.582	Y_{12} 51 28.587
			16	Y_{10} 51 28.466	Y_9 51 28.637	Y_{10} 51 28.623	Y_{11} 51 28.618	Y_{12} 51 28.589
1795	Y_4	5 40 12.974	$w = 1$	... .	... .	Y_6 40 15.065	Y_7 40 10.564	Y_8 40 9.211
	Y_0	5 40 10.397	8	Y_7 40 18.593	Y_9 40 13.726	Y_{10} 40 10.568	Y_{11} 40 10.180	Y_{12} 40 10.219
			16	Y_{10} 40 10.052	Y_9 40 10.389	Y_{10} 40 10.297	Y_{11} 40 10.290	Y_{12} 40 10.227
1787	Y_4	5 29 5.222	$w = 1$	... .	... .	Y_6 29 6.244	Y_7 28 57.334	Y_8 28 54.784
	Y_0	5 28 57.519	8	Y_7 29 11.364	Y_9 29 4.157	Y_{10} 28 58.170	Y_{11} 28 57.106	Y_{12} 28 57.222
			16	Y_{10} 28 56.925	Y_9 28 57.706	Y_{10} 28 57.425	Y_{11} 28 57.416	Y_{12} 28 57.256
1779	Y_4	5 18 10.274	$w = 1$	... .	... .	Y_6 18 8.230	Y_7 17 51.651	Y_8 17 47.106
	Y_0	5 17 52.998	8	Y_7 18 15.075	Y_9 18 5.715	Y_{10} 17 54.895	Y_{11} 17 52.210	Y_{12} 17 52.493
			16	Y_{10} 17 51.794	Y_9 17 53.704	Y_{10} 17 53.035	Y_{11} 17 53.021	Y_{12} 17 52.602
1771	Y_4	5 7 33.100	$w = 1$	... .	... .	Y_6 7 24.642	Y_7 6 55.339	Y_8 6 47.613
	Y_0	5 6 59.542	8	Y_7 7 32.971	Y_9 7 23.095	Y_{10} 7 4.359	Y_{11} 6 58.044	Y_{12} 6 58.616
			16	Y_{10} 6 56.900	Y_9 7 1.459	Y_{10} 7 0.069	Y_{11} 7 0.032	Y_{12} 6 58.938
1763	Y_4	4 57 19.008	$w = 1$	... .	... .	Y_6 56 58.921	Y_7 56 9.323	Y_8 55 56.720
	Y_0	4 56 19.577	8	Y_7 57 7.829	Y_9 57 1 968	Y_{10} 56 30.717	Y_{11} 56 16.736	Y_{12} 56 17.745
			16	Y_{10} 56 13.780	Y_9 56 24.061	Y_{10} 56 21.459	Y_{11} 56 21.328	Y_{12} 56 18.587
1755	Y_4	4 47 33.652	$w = 1$	... .	... .	Y_6 46 54.235	Y_7 45 33.342	Y_8 45 13.502
	Y_0	4 45 55.203	8	Y_7 47 1.745	Y_9 47 9 609	Y_{10} 46 19.216	Y_{11} 45 49.870	Y_{12} 45 51.352
			16	Y_{10} 45 43 018	Y_9 46 4.819	Y_{10} 46 0.346	Y_{11} 45 59.916	Y_{12} 45 53.407

Residuals between the exact Values Y_0 and the Sum of the entire Series $a_0 + U^{\mathrm{I}}t + U^{\mathrm{II}}t^2 \ldots U^{\mathrm{XII}}t^{12}$.

Employing the same notation as on page 280:—

Epoch.	$t - 1875$.		$J = a$		$J = \frac{da}{dt}$		$J = \frac{d^2a}{dt^2}$		$J = \frac{d^3a}{dt^3}$		$J = \frac{d^4a}{dt^4}$	
				s.		s.		s.		s.		s.
1851	− 24	$w=1$	....	....	....	....	Y_0-Y_6	− 0.005	Y_0-Y_7	− 0.005	Y_0-Y_8	− 0.004
		8	Y_0-Y_7	− 0.014	Y_0-Y_9	− 0.006	Y_0-Y_{10}	− 0.003	Y_0-Y_{11}	− 0.004	Y_0-Y_{12}	− 0.004
		16	Y_0-Y_{10}	− 0.002	Y_0-Y_9	− 0.004	Y_0-Y_{10}	− 0.004	Y_0-Y_{11}	− 0.004	Y_0-Y_{12}	− 0.004
1843	− 32	$w=1$	....	....	....	....	Y_0-Y_6	− 0.009	Y_0-Y_7	− 0.005	Y_0-Y_8	− 0.002
		8	Y_0-Y_7	− 0.050	Y_0-Y_9	− 0.012	Y_0-Y_{10}	− 0.002	Y_0-Y_{11}	− 0.001	Y_0-Y_{12}	+ 0.000
		16	Y_0-Y_{10}	+ 0.005	Y_0-Y_9	+ 0.000	Y_0-Y_{10}	+ 0.000	Y_0-Y_{11}	− 0.001	Y_0-Y_{12}	+ 0.000
1835	− 40	$w=1$	....	....	....	....	Y_0-Y_6	− 0.037	Y_0-Y_7	− 0.010	Y_0-Y_8	+ 0.002
		8	Y_0-Y_7	− 0.164	Y_0-Y_9	− 0.037	Y_0-Y_{10}	+ 0.001	Y_0-Y_{11}	+ 0.002	Y_0-Y_{12}	+ 0.003
		16	Y_0-Y_{10}	+ 0.022	Y_0-Y_9	+ 0.005	Y_0-Y_{10}	+ 0.003	Y_0-Y_{11}	+ 0.003	Y_0-Y_{12}	+ 0.004
1827	− 48	$w=1$	....	....	....	....	Y_0-Y_6	− 0.136	Y_0-Y_7	− 0.029	Y_0-Y_8	+ 0.015
		8	Y_0-Y_7	− 0.458	Y_0-Y_9	− 0.111	Y_0-Y_{10}	+ 0.004	Y_0-Y_{11}	+ 0.009	Y_0-Y_{12}	+ 0.010
		16	Y_0-Y_{10}	+ 0.050	Y_0-Y_9	+ 0.012	Y_0-Y_{10}	+ 0.007	Y_0-Y_{11}	+ 0.010	Y_0-Y_{12}	+ 0.012
1819	− 56	$w=1$	....	....	....	....	Y_0-Y_6	− 0.404	Y_0-Y_7	− 0.067	Y_0-Y_8	+ 0.059
		8	Y_0-Y_7	− 1.093	Y_0-Y_9	− 0.294	Y_0-Y_{10}	+ 0.013	Y_0-Y_{11}	+ 0.026	Y_0-Y_{12}	+ 0.027
		16	Y_0-Y_{10}	+ 0.096	Y_0-Y_9	+ 0.030	Y_0-Y_{10}	+ 0.021	Y_0-Y_{11}	+ 0.021	Y_0-Y_{12}	+ 0.028
1811	− 64	$w=1$	....	....	....	....	Y_0-Y_6	− 1.028	Y_0-Y_7	− 0.130	Y_0-Y_8	+ 0.176
		8	Y_0-Y_7	− 2.335	Y_0-Y_9	− 0.710	Y_0-Y_{10}	+ 0.015	Y_0-Y_{11}	+ 0.056	Y_0-Y_{12}	+ 0.059
		16	Y_0-Y_{10}	+ 0.153	Y_0-Y_9	+ 0.050	Y_0-Y_{10}	+ 0.045	Y_0-Y_{11}	+ 0.045	Y_0-Y_{12}	+ 0.060
1803	− 72	$w=1$	....	....	....	....	Y_0-Y_6	− 2.302	Y_0-Y_7	− 0.193	Y_0-Y_8	+ 0.476
		8	Y_0-Y_7	− 4.545	Y_0-Y_9	− 1.586	Y_0-Y_{10}	− 0.016	Y_0-Y_{11}	+ 0.113	Y_0-Y_{12}	+ 0.108
		16	Y_0-Y_{10}	+ 0.229	Y_0-Y_9	+ 0.058	Y_0-Y_{10}	+ 0.072	Y_0-Y_{11}	+ 0.077	Y_0-Y_{12}	+ 0.106
1795	− 80	$w=1$	....	....	....	....	Y_0-Y_6	− 4.668	Y_0-Y_7	− 0.167	Y_0-Y_8	+ 1.186
		8	Y_0-Y_7	− 8.196	Y_0-Y_9	− 3.329	Y_0-Y_{10}	− 0.171	Y_0-Y_{11}	+ 0.217	Y_0-Y_{12}	+ 0.178
		16	Y_0-Y_{10}	+ 0.345	Y_0-Y_9	+ 0.008	Y_0-Y_{10}	+ 0.100	Y_0-Y_{11}	+ 0.107	Y_0-Y_{12}	+ 0.170
1787	− 88	$w=1$	....	....	....	....	Y_0-Y_6	− 8.725	Y_0-Y_7	+ 0.185	Y_0-Y_8	+ 2.735
		8	Y_0-Y_7	−13.845	Y_0-Y_9	− 6.638	Y_0-Y_{10}	− 0.651	Y_0-Y_{11}	+ 0.413	Y_0-Y_{12}	+ 0.297
		16	Y_0-Y_{10}	+ 0.594	Y_0-Y_9	− 0.187	Y_0-Y_{10}	+ 0.094	Y_0-Y_{11}	+ 0.103	Y_0-Y_{12}	+ 0.263
1779	− 96	$w=1$	....	....	....	....	Y_0-Y_6	−15.232	Y_0-Y_7	+ 1.347	Y_0-Y_8	+ 5.892
		8	Y_0-Y_7	−22.077	Y_0-Y_9	−12.717	Y_0-Y_{10}	− 1.897	Y_0-Y_{11}	+ 0.788	Y_0-Y_{12}	+ 0.505
		16	Y_0-Y_{10}	+ 1.204	Y_0-Y_9	− 0.706	Y_0-Y_{10}	− 0.037	Y_0-Y_{11}	− 0.023	Y_0-Y_{12}	+ 0.396
1771	−104	$w=1$	....	....	....	....	Y_0-Y_6	−25.100	Y_0-Y_7	+ 4.203	Y_0-Y_8	+11.929
		8	Y_0-Y_7	−33.429	Y_0-Y_9	−23.553	Y_0-Y_{10}	− 4.817	Y_0-Y_{11}	+ 1.498	Y_0-Y_{12}	+ 0.926
		16	Y_0-Y_{10}	+ 2.642	Y_0-Y_9	− 1.917	Y_0-Y_{10}	− 0.527	Y_0-Y_{11}	− 0.490	Y_0-Y_{12}	+ 0.604
1763	−112	$w=1$	....	....	....	....	Y_0-Y_6	−39.344	Y_0-Y_7	+10.254	Y_0-Y_8	+22.857
		8	Y_0-Y_7	−48.252	Y_0-Y_9	−42.391	Y_0-Y_{10}	−11.140	Y_0-Y_{11}	+ 2.841	Y_0-Y_{12}	+ 1.832
		16	Y_0-Y_{10}	+ 5.797	Y_0-Y_9	− 4.484	Y_0-Y_{10}	− 1.882	Y_0-Y_{11}	− 1.751	Y_0-Y_{12}	+ 0.990
1755	−120	$w=1$	....	....	....	....	Y_0-Y_6	−59.032	Y_0-Y_7	+21.861	Y_0-Y_8	+41.701
		8	Y_0-Y_7	−66.542	Y_0-Y_9	−74.406	Y_0-Y_{10}	−24.013	Y_0-Y_{11}	+ 5.333	Y_0-Y_{12}	+ 3.851
		16	Y_0-Y_{10}	+12.185	Y_0-Y_9	− 9.616	Y_0-Y_{10}	− 5.143	Y_0-Y_{11}	− 4.713	Y_0-Y_{12}	+ 1.796

The following conclusions are drawn from an examination of these residuals. They relate strictly to the star Groombridge 1119, but they will apply in a general way to all stars having nearly the same declination.

(*a*) Between $t = 0$ and $t = 32$ the correspondence between the results obtained from the Bohnenberger equations and those found by the development by Taylor's Theorem is sufficiently exact, whatever quantity is taken as the initial function in the computation of the differential coefficients, and whatever value is given to w, except when the initial function is α.

(*b*) For any date earlier than 1835 the development fails when $w = 1$, while the limit when $w = 8$ may be placed at about 1825.

(*c*) Between the limits $w = 1$ and $w = 16$ every form of development fails when t exceeds fifty years. From this point the magnitude of the residuals varies with the choice of the quantity taken as the initial function, and with the value of w.

(*d*) Variations in the value of w produce the least effect when $J = \frac{d^4 \alpha}{d t^4}$, and the greatest effect when $J = \alpha$.

(*e*) Whatever quantity is taken as the initial function in the computation of the differential coefficients, there is a substantial agreement in the values of the residuals between $t = 0$ and $t = 50$ when $w = 16$.

(*f*) When α is taken as the initial function, w should never be taken less than 16 when t exceeds 20 years, and a new equinox should be chosen when t exceeds 40 years.

(*g*) Between $t = 0$ and $t = 80$, $\frac{d\alpha}{dt}$ may be advantageously taken as the initial function in the computation of the differential coefficients, both on account of the smallness of the residuals for $w = 16$, and on account of the comparatively trifling labor involved in the computation.

(*h*) Notwithstanding the general increase in the accuracy of the development with an increase in the number of the terms of the series, the gain when t exceeds 40 years is so slight that it will be better in any case to change the equinox at intervals of 30 years. In computing the new coefficients, $\frac{d\alpha}{dt}$ may be advantageously selected as the initial function. It will be sufficient to carry the computation to the sixth term inclusive if $w = 16$.

(*i*) It will be seen, therefore, that the advantage gained by the increase in the number of terms of the series may be counterbalanced by the effect of the

unavoidable errors in the logarithmic computation of the initial functions. Hence, it may happen that it will be a positive disadvantage to extend the development beyond a certain limit. This limit appears, in the present case, to be near the seventh term when $w = 16$ and $t = 50$.

Development of the Functions α and δ by Mechanical Quadratures.

The differential coefficients already computed hold true only for the instant of time at which the initial functions α and δ are assumed to be true. In the series of differences obtained from these functions, the differential coefficients all fall upon the same horizontal line.

In order to obtain the summed series which will represent the values of α and δ at any assumed epoch of the constants of the precession, it will be necessary to find the values of the differential coefficients which correspond to the instant $t + \frac{1}{2}$ or $t - \frac{1}{2}$.

If the series does not extend beyond 24 years, the fourth term may be taken as a constant. The coefficients may be converted into differences from which the summation may be directly made by the following relations:

Functions for the Instant t_0.	Equivalent Functions for the Instant $t \pm \frac{1}{2}$.
$\Delta_4 = \dfrac{d^4\alpha}{dt^4} = \Delta^{\mathrm{IV}}$	$\Delta_4 = \Delta^{\mathrm{IV}} = \alpha$ constant
$\Delta_3 = \dfrac{d^3\alpha}{dt^3} =$	$\Delta_3 = \Delta^{\mathrm{III}}$
$\Delta_2 = \dfrac{d^2\alpha}{dt^2} = \Delta^{\mathrm{II}}$	$\Delta_2 = \Delta^{\mathrm{II}} + \frac{1}{12}\Delta^{\mathrm{IV}}$
$\Delta_1 = \dfrac{d\alpha}{dt} = \Delta^{\mathrm{I}}$	$\Delta_1 = \Delta^{\mathrm{I}} + \frac{1}{6}\Delta^{\mathrm{III}}$
	$\Delta_1 + \frac{1}{2} = \Delta_1 + \frac{1}{2}\Delta_2$

From the example given on page 247 we have the following data:

Epoch.	α	$\dfrac{d\alpha}{dt}$	$\dfrac{d^2\alpha}{dt^2}$	$\dfrac{d^3\alpha}{dt^3}$	$\dfrac{d^4\alpha}{dt^4}$
	h. m. s.	s.	s.	s.	s.
1875.0	7 29 5.631000	73.51432400	−0.30237888	−0.00203746	+0.00008334

Whence for the instant $t \pm \frac{1}{2}$

$$\begin{aligned}
\Delta_4 &= +\ \overset{s.}{0.00008334} \\
\Delta_3 &= -\ 0.00203746 \\
\Delta_2 &= -\ 0.30237888 + \frac{\overset{s.}{0.00008334}}{12} = -\ \overset{s.}{0.30237194} \\
\Delta_1 &= +73.51432400 - \frac{0.00203746}{6} = +73.51398442
\end{aligned}$$

We have therefore:

Epoch.	α	Δ_1	Δ_2	Δ_3	Δ_4
	h. m. s.	s.	s.	s.	s.
1875.0	7 29 5.63100000	+73.51398442	−0.30237194	−0.00203746	+0.00008334
		+73.36279845		−0.00199580	
1876.0	7 30 18.99379845		−0.30436774		

The summation in either direction from 1875 may now be continued by the successive additions or subtractions of the differences.

The application of this method will be illustrated by the computation of α for each year from 1875 to 1899.

Epoch.	α	Δ_1	Δ_2	Δ_3	Δ_4
	h. m. s.				
1875	7 29 5.63100000				+0.00008334
		+73.36279845		−0.00199580	
1876	7 30 18.99379845		−0.30436774		+0.00008334
		+73.05843071		−0.00191246	
1877	7 31 32.05222916		−0.30628020		+0.00008334
		+72.75215051		−0.00182912	
1878	7 32 44.80437967		−0.30810932		+0.00008334
		+72.44404119		−0.00174578	
1879	7 33 57.24842086		−0.30985510		+0.00008334
		+72.13418609		−0.00166244	
1880	7 35 9.38260695		−0.31151754		+0.00008334
		+71.82266855		−0.00157910	
1881	7 36 21.20527550		−0.31309664		+0.00008334
		+71.50957191		−0.00149576	
1882	7 37 32.71484741		−0.31459240		+0.00008334
		+71.19497951		−0.00141242	
1883	7 38 43.90982692		−0.31600482		+0.00008334
		+70.87897469		−0.00132908	
1884	7 39 54.78880161		−0.31733390		+0.00008334
		+70.56164079		−0.00124574	
1885	7 41 5.35044240		−0.31857964		+0.00008334
		+70.24306115		−0.00116240	
1886	7 42 15.59350355		−0.31974204		+0.00008334
		+69.92331911		−0.00107906	
1887	7 43 25.51682266		−0.32082110		+0.00008334
		+69.60249801		−0.00099572	

Epoch.	α	Δ_1	Δ_2	Δ_3	Δ_4
	h. m. s.	+69.60249801		−0.00099572	
1888	7 44 35.11932067		−0.32181682		+0.00008334
		+69.28068119		−0.00091238	
1889	7 45 44.40000186		−0.32272920		+0.00008334
		+68.95795199		−0.00082904	
1890	7 46 53.35795385		−0.32355824		+0.00008334
		+68.63439375		−0.00074570	
1891	7 48 1.99234750		−0.32430394		+0.00008334
		+68.31008981		−0.00066236	
1892	7 49 10.30243731		−0.32496630		+0.00008334
		+67.98512351		−0.00057902	
1893	7 50 18.28756082		−0.32554532		+0.00008334
		+67.65957819		−0.00049568	
1894	7 51 25.94713901		−0.32604100		+0.00008334
		+67.33353719		−0.00041234	
1895	7 52 33.28067620		−0.32645334		+0.00008334
		+67.00708385		−0.00032900	
1896	7 53 40.28776005		−0.32678234		+0.00008334
		+66.68030151		−0.00024566	
1897	7 54 46.96806156		−0.32702800		+0.00008334
		+66.35327351		−0.00016232	
1898	7 55 53.32133507		−0.32719032		+0.00008334
		+66.02608319		−0.00007898	
1899	7 56 59.34741826		−0.32726930		

It will be seen that the result for 1899 agrees with that already found to the fourth decimal place.

Reduction of α and δ from any Time t_0 to any Time t' by Means of the Precession for the Mean Interval.

Main has given the following expressions (Mem. Roy. Astron. Soc., XIX. pp. 127, 128, 1851), which he has derived from the method printed out by O. Struve in his "Bestimmung der Constante der Præcession" (Mém. de l'Acad. Imp. de St. Petersburgh, 6th Series, III. p. 49, 1841).

Let p_1, p_2, p_3, p_4, p_5, represent the values of the precessions for the times t, $2t$, $3t$, $4t$, $5t$, and let P_3, P_4, and P_5 represent the precessions for the mean intervals.

Then for the total effect of the precession at the end of any given time t, we have:

For 3 equidistant intervals $\alpha = \alpha_0 + \frac{1}{6}\,[p_1 + 4\,p_2 + p_3]\,t = \alpha_0 + P_3\,t$ (54)

For 4 equidistant intervals $\alpha = \alpha_0 + \frac{1}{8}\,[p_1 + 3\,p_2 + 3\,p_3 + p_4]\,t = \alpha_0 + P_4\,t$ (55)

For 5 equidistant intervals $\alpha = \alpha_0 + \frac{1}{90}\,[7\,p_1 + 32\,p_2 + 12\,p_3 + 32\,p_4 + 7\,p_5]\,t = \alpha_0 + P_5\,t$ (56)

And similarly for δ.

The values of $p_1 \ldots p_5$ in the application of this method should, in practice, be computed from the co-ordinates derived directly from the observations. If, however, there are not a sufficient number of observations available for the formation of the equal intervals, the first three or four values of α derived by the method already described may be used for the computation of the corresponding values of p.

After this the values of p can be carried forward by differences with sufficient accuracy for the computation of the values of α and δ in advance of the last value computed. It will be necessary to carry the reductions for α and δ along together.

For the computation of the values p_1, p_2, p_3, &c., equations (16) and (20) become:

$$\frac{d\,\alpha}{d\,t} = (m + m'\,t) + (n + n'\,t)\,\sin\alpha\,\tan\delta \tag{57}$$

$$\frac{d\,\delta}{d\,t} = (n + n'\,t)\,\cos\alpha \tag{58}$$

The values of $(m + m'\,t)$ and $\log\,(n + n'\,t)$ can be conveniently written in tabular form. The values for intervals of four years from 1875 to 1899 are as follows:—

Epoch.	$[m + m'\,t]$	$[\log n + n'\,t]$	$\log\,[n + n'\,t]$
	s.	s.	$''$
1875	3.0722450	0.1261147	1.3022060
1879	3.0723209	0.1261075	1.3021988
1883	3.0723969	0.1261003	1.3021916
1887	3.0724729	0.1260931	1.3021844
1891	3.0725488	0.1260859	1.3021772
1895	3.0726248	0.1260787	1.3021700
1899	3.0727008	0.1260715	1.3021628
	&c.	&c.	&c.

In illustration of the application of this method of transferring the co-ordinates α and δ for any epoch t_0 to any epoch t', we select as provisional values of α and δ those derived by computation from the differential coefficients to the fourth term inclusive, from 1875 to 1899. They are given on page 268. The corresponding values of p_1, p_2, p_3, have been computed from equations (57) and (58).

	α			$\frac{d\alpha}{dt}$	δ			$\frac{d\delta}{dt}$
	h.	m.	s.	s.	°	′	″	″
1875	7	29	5.631	+ 73.514324	+ 88	59	37.69	− 7.601
1879	7	33	57.248	+ 72.289466	+ 88	59	6.50	− 7.993
1883	7	38	43.910	+ 71.037178	+ 88	58	33.76	− 8.375
1887	7	43	25.517	+ 69.762627	+ 88	57	59.52	− 8.746
1891	7	48	1.992	+ 68.470504	+ 88	57	23.81	− 9.107
1895	7	52	33.281	+ 67.165522	+ 88	56	46.68	− 9.458
1899	7	56	59.347	+ 65.851860	+ 88	56	8.17	− 9.798

We now compute the values of α for the years 1883 to 1899 by formulæ (54):

		s.
For 1875	$p_1 =$	+ 73.514324
1879	$4p_2 =$	+289.157864
1883	$p_3 =$	+ 71.037178
	Sum	+433.709366

		h.	m.	s.
	$P_3 =$	+0	0	72.284894
	$8P_3 =$	+0	9	38.2791
	$\alpha_0 =$	7	29	5.6310
For 1883	$\alpha =$	7	38	43.9101

		s.
For 1879	$p_1 =$	+ 72.289466
1883	$4p_2 =$	+284.148712
1887	$p_3 =$	+ 69.762627
	Sum	+426.200805

		h.	m.	s.
	$P_3 =$	+0	0	71.033468
	$8P_3 =$	+0	9	28.2677
	$\alpha_0 =$	7	33	57.2480
For 1887	$\alpha =$	7	43	25.5157

		s.
For 1883	$p_1 =$	+ 71.037178
1887	$4p_2 =$	+279.050508
1891	$p_3 =$	+ 68.470504
	Sum	+418.558190

		h.	m.	s.
	$P_3 =$	+0	0	69.759698
	$8P_3 =$	+0	9	18.0776
	$\alpha_0 =$	7	38	43.9101
For 1891	$\alpha =$	7	48	1.9877

		s.
For 1887	$p_1 =$	+ 69.762627
1891	$4p_2 =$	+273.882016
1895	$p_3 =$	+ 67.165522
	Sum	+410.810165

		h.	m.	s.
	$P_3 =$	+0	0	68.468361
	$8P_3 =$	+0	9	7.7469
	$\alpha_0 =$	7	43	25.5157
For 1895	$\alpha =$	7	52	33.2626

From this point the approximate values of $\frac{d\alpha}{dt}$ required for the computation of the next and the following dates can be found from the successive orders of differences of $\frac{d\alpha}{dt}$ already found. Each result for α should be verified by a second computation before proceeding to the next in order. The values of the precessions in declination vary so slowly that a second computation will not be necessary.

We now have:

	α	$\frac{d\alpha}{dt}$	Δ_1	Δ_2	Δ_3	δ	$\frac{d\delta}{dt}$	Δ_1	Δ_2
	h. m. s.	s.				° ′ ″	″		
1875	7 29 5.6310	+73.5143				+88 59 37.69	−7.601		
			−1.2248					−0.392	
1879	7 33 57.2480	+72.2895		−0.0275		+88 59 6.50	−7.993		+0.010
			−1.2523		+0.0052			−0.382	
1883	7 38 43.9101	+71.0372		−0.0223		+88 58 33.76	−8.375		+0.011
			−1.2746		+0.0048			−0.371	
1887	7 43 25.5157	+69.7626		−0.0175		+88 57 59.52	−8.746		+0.010
			−1.2921		+0.0046			−0.361	
1891	7 48 1.9877	+68.4705		−0.0129		+88 57 23.81	−9.107		+0.010
			−1.3050					−0.351	
1895	7 52 33.2626	+67.1655				+88 56 46.68	−9.458		

Assuming $+0^s.0044$ for the third difference of $\frac{d\alpha}{dt}$ opposite 1893, and $+0''.010$ for the second difference of $\frac{d\delta}{dt}$, we have:

	α	$\frac{d\alpha}{dt}$	Δ_1	Δ_2	Δ_3	δ	$\frac{d\delta}{dt}$	Δ_1	Δ_2
	h. m. s.	s.				° ′ ″	″		
1891	7 48 1.9877	+68.4705		−0.0129		+88 57 23.81	− 9.107		+0.010
			−1.3050		+0.0044			−0.351	
1895	7 52 33.2626	+67.1655		−0.0085		+88 56 46.68	− 9.458		+0.011
			−1.3135		+0.0042			−0.340	
1899	7 56 59.3003	+65.8520		−0.0043		+88 56 8.17	− 9.798		+0.010
			−1.3178		+0.0040			−0.330	
1903	8 1 20.0768	+64.5342		−0.0003		+88 55 28.32	−10.128		+0.010
			−1.3181					−0.320	
1907	8 5 35.7067	+63.2161				+88 54 47.17	−10.448		

After proceeding thus far, it will be advisable to recompute the values of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$, and the resulting values of α and δ. Then, with the more accurate series of differences, the computation for several intervals in advance can be safely made. It will be found that the values of δ and $\frac{d\delta}{dt}$ require no change. For α and $\frac{d\alpha}{dt}$ we find:

Epoch.	α			$\frac{d\alpha}{dt}$	Δ_1	Δ_2	Δ_3
	h.	m.	s.	s.	s.	s.	s.
1875	7	29	5.6310	+73.51432			
					−1.22485		
1879	7	33	57.2480	+72.28947		−0.02744	
					−1.25229		+0.00515
1883	7	38	43.9101	+71.03718		−0.02229	
					−1.27458		+0.00480
1887	7	43	25.5157	+69.76260		−0.01749	
					−1.29207		+0.00460
1891	7	48	1.9877	+68.47053		−0.01289	
					−1.30496		+0.00425
1895	7	52	33.2626	+67.16557		−0.00864	
					−1.31360		+0.00406
1899	7	56	59.3003	+65.85197		−0.00458	
					−1.31818		+0.00369
1903	8	1	20.0709	+64.53379		−0.00089	
					−1.31907		
1907	8	5	35.5693	+63.21472			

As a further illustration of the application of formulæ (54), (55), and (56), we compute the value of α for 1911 from the values of $\frac{d\alpha}{dt}$ for 1895, 1903, 1911, by formula (54); for 1887, 1895, 1903, and 1911, by formula (55); and for 1879, 1887, 1895, 1903, and 1911, by formula (56). The value of $\frac{d\alpha}{dt}$ for 1911 is carried forward from that for 1907 by means of the third difference $+0^s.00341$, giving $\frac{d\alpha}{dt} = +61^s.89817$.

$$
\begin{array}{llrrr}
1895 & p_1 = & & & +\ 67^s.16557 \\
1903 & 4p_2 = & & & +258.13516 \\
1911 & p_3 = & & & +\ 61.89817 \\
 & \text{Sum} = & & & +387.19890 \\
 & P_3 = & & & +\ 64.53315 \\
 & & h. & m. & s. \\
 & 16P_3 = & +0 & 17 & 12.5304 \\
 & \alpha_0 = & 7 & 52 & 33.2626 \\
\text{For 1911} & \alpha = & 8 & 9 & 45.7930
\end{array}
$$

$$
\begin{array}{llrrr}
1887 & p_1 = & & & +\ 69^s.76260 \\
1895 & 3p_2 = & & & +201.49671 \\
1903 & 3p_3 = & & & +193.60137 \\
1911 & p_4 = & & & +\ 61.89817 \\
 & \text{Sum} = & & & +526.75885 \\
 & P_4 = & & & +\ 65.84486 \\
 & & h. & m. & s. \\
 & 24P_4 = & +0 & 26 & 20.2766 \\
 & \alpha_0 = & 7 & 43 & 25.5157 \\
\text{For 1911} & \alpha = & 8 & 9 & 45.7923
\end{array}
$$

$$
\begin{array}{llrrr}
1879 & 7p_1 = & & & +\ 506^s.02629 \\
1887 & 32p_2 = & & & +2232.40320 \\
1895 & 12p_3 = & & & +\ 805.98684 \\
1903 & 32p_4 = & & & +2065.08128 \\
1911 & 7p_5 = & & & +\ 433.28719 \\
 & \text{Sum} = & & & +6042.78480 \\
 & P_5 = & & & +\ 67.14205 \\
 & & h. & m. & s. \\
 & 32P_5 = & +0 & 35 & 48.5457 \\
 & \alpha_0 = & 7 & 33 & 57.2480 \\
\text{For 1911} & \alpha = & 8 & 9 & 45.7937
\end{array}
$$

Development of the Functions α and δ by Means of two or more Partial Series expressed in Terms of the Ascending Powers of the Time.

It has been shown that it is impossible to obtain the exact development of the primary functions for stars within one degree of the pole when the time exceeds forty years, even in the most favorable case which can occur.

The time at which the values of the initial functions derived from the development by Taylor's Theorem begin to deviate from those derived from equations (6) may be extended many years by means of a secondary series which represents the residuals between the exact co-ordinates and those obtained with any assumed limit to the terms of the series.

Let $Y_0 =$ the values of the functions α or δ derived from equations (6).

$Y_3, Y_4, Y_5, \ldots Y_{12} =$ the values of α or δ derived from the development which terminates with the 3d, 4th, 5th to the 12th term.

We shall then have a series of residuals $Y_0 - Y_3$, $Y_0 - Y_4$, $Y_0 - Y_5$, . . . $Y_0 - Y_{12}$, any one of which may be represented by a series similar in form to the primary series from which the residuals have been obtained.

It is to be remarked, however, that this second series will not be continuous with respect to the first.

If these residuals are obtained for the equidistant intervals w, $2w$, $3w$, $4w$, &c. years, we shall have a series of equations in which the number of the equations is the same as the number of the unknown quantities. We shall thus obtain an exact representation of the residuals for the intervals chosen, and a close approximation to the true values for any intermediate interval which does not much exceed twenty years.

If the residuals really follow the law expressed by Taylor's Theorem, we shall also be able to extend the agreement considerably beyond the limit for the largest value of $n\,w$ employed, n being the coefficient of w.

Let
$a =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for w years,
$b =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for $2w$ years,
$c =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for $3w$ years,
$d =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for $4w$ years,
$e =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for $5w$ years,
$f =$ the value of $Y_0 - (Y_3 \ldots Y_{12})$ for $6w$ years, &c.

We shall have by Taylor's Theorem:

For Three Equidistant Values of w.

If $\frac{d' \alpha}{d t} = x, \quad \frac{d'^2 \alpha}{d t^2} = y,$ and $\frac{d'^3 \alpha}{d t^3} = z,$

$$\begin{aligned} w x + \frac{1}{2} w^2 y + \frac{1}{6} w^3 z &= a \\ w x + w^2 y + \frac{2}{3} w^3 z &= \frac{b}{2} \\ w x + \frac{3}{2} w^2 y + \frac{3}{2} w^3 z &= \frac{c}{3} \end{aligned} \tag{59}$$

Whence

$$\begin{aligned} x &= \frac{1}{w} [3 a - \frac{3}{2} b + \frac{1}{3} c] \\ y &= \frac{1}{w^2} [4 b - 5 a - c] \\ z &= \frac{1}{w^3} [3 a - 3 b + c] \end{aligned} \tag{60}$$

For Four Equidistant Values of w.

If $\frac{d' \alpha}{d t} = v, \quad \frac{d'^2 \alpha}{d t^2} = x, \quad \frac{d'^3 \alpha}{d t^3} = y, \quad \frac{d'^4 \alpha}{d t^4} = z,$ we have

$$\begin{aligned} w v + \frac{1}{2} w^2 x + \frac{1}{6} w^3 y + \frac{1}{24} w^3 z &= a \\ w v + w^2 x + \frac{2}{3} w^3 y + \frac{1}{3} w^4 z &= \frac{b}{2} \\ w v + \frac{3}{2} w^2 x + \frac{3}{2} w^3 y + \frac{9}{8} w^4 z &= \frac{c}{3} \\ w v + 2 w^2 x + \frac{8}{3} w^3 y + \frac{8}{3} w^4 z &= \frac{d}{4} \end{aligned} \tag{61}$$

Whence

$$\begin{aligned} v &= \frac{1}{w} \left[4 a - 3 b + \frac{4}{3} c - \frac{1}{4} d\right] \\ x &= \frac{1}{w^2} \left[\frac{11}{12} d + \frac{19}{2} b - \frac{14}{3} c - \frac{26}{3} a\right] \\ y &= \frac{1}{w^3} \left[9 a - 12 b + 7 c - \frac{3}{2} d\right] \\ z &= \frac{1}{w^4} \left[d - 4 c + 6 b - 4 a\right] \end{aligned} \tag{62}$$

For Five Equidistant Values of w.

If $\frac{d' \alpha}{d t} = u, \quad \frac{d'^2 \alpha}{d t^2} = v, \quad \frac{d'^3 \alpha}{d t^3} = x, \quad \frac{d'^4 \alpha}{d t^4} = y, \quad \frac{d'^5 \alpha}{d t^5} = z,$ we have

$$
\left.\begin{aligned}
& w u + \frac{1}{2} w^2 v + \frac{1}{6} w^3 x + \frac{1}{24} w^4 y + \frac{1}{120} w^5 z = a \\
& w u + w^2 v + \frac{2}{3} w^3 x + \frac{1}{3} w^4 y + \frac{2}{15} w^5 z = \frac{b}{2} \\
& w u + \frac{3}{2} w^2 v + \frac{3}{2} w^3 x + \frac{9}{8} w^4 y + \frac{27}{40} w^5 z = \frac{c}{3} \\
& w u + 2 w^2 v + \frac{8}{3} w^3 x + \frac{8}{3} w^4 y + \frac{32}{15} w^5 z = \frac{d}{4} \\
& w u + \frac{5}{2} w^2 v + \frac{25}{6} w^3 x + \frac{125}{24} w^4 y + \frac{125}{24} w^5 z = \frac{e}{5}
\end{aligned}\right\} \tag{63}
$$

Whence

$$
\left.\begin{aligned}
u &= \frac{1}{w} \left[5 a - 5 b + \frac{10}{3} c - \frac{5}{4} d + \frac{e}{5}\right] \\
v &= \frac{1}{w^2} \left[\frac{107}{6} b - \frac{77}{6} a - 13 c + \frac{61}{12} d - \frac{5}{6} e\right] \\
x &= \frac{1}{w^3} \left[\frac{71}{4} a - \frac{59}{2} b + \frac{49}{2} c - \frac{41}{4} d + \frac{7}{4} e\right] \\
y &= \frac{1}{w^4} \left[26 b - 14 a - 24 c + 11 d - 2 e\right] \\
z &= \frac{1}{w^5} \left[5 a - 10 b + 10 c - 5 d + e\right]
\end{aligned}\right\} \tag{64}
$$

For Six Equidistant Values of w.

If $\frac{d' \alpha}{d t} = s, \quad \frac{d'^2 \alpha}{d t^2} = u, \quad \frac{d'^3 \alpha}{d t^3} = v, \quad \frac{d'^4 \alpha}{d t^4} = x, \quad \frac{d'^5 \alpha}{d t^5} = y, \quad \frac{d'^6 \alpha}{d t^6} = z,$ we have

$$
\left.\begin{aligned}
& w s + \frac{1}{2} w^2 u + \frac{1}{6} w^3 v + \frac{1}{24} w^4 x + \frac{1}{120} w^5 y + \frac{1}{720} w^6 z = a \\
& w s + w^2 u + \frac{2}{3} w^3 v + \frac{1}{3} w^4 x + \frac{2}{15} w^5 y + \frac{2}{45} w^6 z = \frac{b}{2} \\
& w s + \frac{3}{2} w^2 u + \frac{3}{2} w^3 v + \frac{9}{8} w^4 x + \frac{27}{40} w^5 y + \frac{27}{80} w^6 z = \frac{c}{3} \\
& w s + 2 w^2 u + \frac{8}{3} w^3 v + \frac{8}{3} w^4 x + \frac{32}{15} w^5 y + \frac{64}{45} w^6 z = \frac{d}{4} \\
& w s + \frac{5}{2} w^2 u + \frac{25}{6} w^3 v + \frac{125}{24} w^4 x + \frac{125}{24} w^5 y + \frac{625}{144} w^6 z = \frac{e}{5} \\
& w s + 3 w^2 u + 6 w^3 v + 9 w^4 x + \frac{54}{5} w^5 y + \frac{54}{5} w^6 z = \frac{f}{6}
\end{aligned}\right\} \tag{65}
$$

Whence

$$\left.\begin{aligned}
s &= \frac{1}{w}\left[6\,a - \frac{15}{2}\,b + \frac{20}{3}\,c - \frac{15}{4}\,d + \frac{6}{5}\,e - \frac{1}{6}\,f\right] \\
u &= \frac{1}{w^2}\left[\frac{117}{4}\,b - \frac{87}{5}\,a - \frac{254}{9}\,c + \frac{33}{2}\,d - \frac{27}{5}\,e + \frac{137}{180}\,f\right] \\
v &= \frac{1}{w^3}\left[29\,a - \frac{461}{8}\,b + 62\,c - \frac{307}{8}\,d + 13\,e - \frac{15}{8}\,f\right] \\
x &= \frac{1}{w^4}\left[\frac{137}{2}\,b - 31\,a - \frac{242}{3}\,c + \frac{107}{2}\,d - 19\,e + \frac{17}{6}\,f\right] \\
y &= \frac{1}{w^5}\left[20\,a - \frac{95}{2}\,b + 60\,c - \frac{85}{2}\,d + 16\,e - \frac{5}{2}\,f\right] \\
z &= \frac{1}{w^6}\left[15\,b - 6\,a - 20\,c + 15\,d - 6\,e + f\right]
\end{aligned}\right\} \quad (66)$$

The final equation for α will then become:

$$\begin{aligned}
\alpha = \alpha_0 &+ \frac{d\,\alpha}{d\,t}\,t + \frac{1}{2}\,\frac{d^2\,\alpha}{d\,t^2}\,t^2 + \frac{1}{6}\,\frac{d^3\,\alpha}{d\,t^3}\,t^3 + \frac{1}{24}\,\frac{d^4\,\alpha}{d\,t^4}\,t^4 + \frac{1}{120}\,\frac{d^5\,\alpha}{d\,t^5}\,t^5 + \frac{1}{720}\,\frac{d^6\,\alpha}{d\,t^6}\,t^6, \text{ \&c.} \\
&+ \frac{d'\,\alpha}{d\,t}\,t + \frac{1}{2}\,\frac{d'^2\,\alpha}{d\,t^2}\,t^2 + \frac{1}{6}\,\frac{d'^3\,\alpha}{d\,t^3}\,t^3 + \frac{1}{24}\,\frac{d'^4\,\alpha}{d\,t^4}\,t^4 + \frac{1}{120}\,\frac{d'^5\,\alpha}{d\,t^5}\,t^5 + \frac{1}{720}\,\frac{d'^6\,\alpha}{d\,t^6}\,t^6, \text{ \&c.} \\
= \alpha_0 &+ \left(\frac{d\,\alpha}{d\,t} + \frac{d'\,\alpha}{d\,t}\right) + \frac{1}{2}\left(\frac{d^2\,\alpha}{d\,t^2} + \frac{d'^2\,\alpha}{d\,t^2}\right)t^2 + \frac{1}{6}\left(\frac{d^3\,\alpha}{d\,t^3} + \frac{d'^3\,\alpha}{d\,t^3}\right)t^3 + \frac{1}{24}\left(\frac{d^4\,\alpha}{d\,t^4} + \frac{d'^4\,\alpha}{d\,t^4}\right), \text{\&c.,}
\end{aligned}$$

and similarly for δ.

As an illustration of the application of this method, we select the values of $Y_0 - Y_4$ given on page 267. For intervals of sixteen years from 1875 to 1779 we have:

Epoch.	t	$Y_0 - Y_4$	Epoch.	t	$Y_0 - Y_4$
		s.			s.
1859	16	$a = -0.001$	1811	64	$d = +\ 0.612$
1843	32	$b = +0.102$	1795	80	$e = -\ 2.577$
1827	48	$c = +0.480$	1779	96	$f = -17.276$

Whence by equations (66)

s.	s.	s.
$+\ 6\ a = -0.00600$	$+\frac{117}{4}\,b = +\ 2.98350$	$+\ 29\ a = -\ 0.02900$
$-\frac{15}{2}\,b = -0.76500$	$-\ \frac{87}{5}\ a = +\ 0.01740$	$-\frac{461}{8}\,b = -\ 5.87778$
$+\frac{20}{3}\,c = +3.20000$	$-\frac{254}{9}\,c = -13.54667$	$+\ 62\ c = +29.76000$
$-\frac{15}{4}\,d = -2.29500$	$+\ \frac{33}{2}\ d = +10.09800$	$-\frac{307}{8}\,d = -23.48550$
$+\ \frac{6}{5}\ e = -3.09240$	$-\ \frac{27}{5}\ e = +13.91580$	$+\ 13\ e = -33.50100$
$-\ \frac{1}{6}\ f = +2.87933$	$+\frac{137}{180}\,f = -13.14896$	$-\ \frac{15}{8}\ f = +32.39250$
Sum $= -0.07907$	Sum $= +\ 0.31907$	Sum $= -\ 0.74078$

log sum	8.89801n
log 16	1.20412
log s	7.69389
log 1	0.00000
$\log \frac{d' \alpha}{d t}$	7.69389n

$$\frac{d' \alpha}{d t} = -0.0049419$$

log sum	9.50389
log 16^2	2.40824
log u	7.09565
log 2	0.30103
$\log \frac{1}{2} \frac{d'^2 \alpha}{d^2 t}$	6.79462

$$\frac{1}{2} \frac{d'^2 \alpha}{d t^2} = +0.00062318$$

log sum	9.86969n
log 16^3	3.61236
log v	6.25733n
log 6	0.77815
$\log \frac{1}{6} \frac{d'^3 \alpha}{d t^3}$	5.47918n

$$\frac{1}{6} \frac{d'^3 \alpha}{d t^3} = -0.000030143$$

$$\begin{aligned} +\tfrac{137}{2} b &= + \overset{s.}{6.98700} \\ -31\, a &= + 0.03100 \\ -\tfrac{242}{3} c &= -38.72000 \\ +\tfrac{107}{2} d &= +32.74200 \\ -19\, e &= +48.96300 \\ +\tfrac{17}{6} f &= \underline{-48.94868} \\ \text{Sum} &= + 1.05432 \end{aligned}$$

$$\begin{aligned} +20\, a &= - \overset{s.}{0.02000} \\ -\tfrac{95}{2} b &= - 4.84500 \\ +60\, c &= +28.80000 \\ -\tfrac{85}{2} d &= -26.01000 \\ +16\, e &= -41.23200 \\ -\tfrac{5}{2} f &= \underline{+43.19000} \\ \text{Sum} &= - 0.11700 \end{aligned}$$

$$\begin{aligned} +15\, b &= + \overset{s.}{1.530} \\ -6\, a &= + 0.006 \\ -20\, c &= - 9.600 \\ +15\, d &= + 9.180 \\ -6\, e &= +15.462 \\ + f &= \underline{-17.276} \\ \text{Sum} &= - 0.698 \end{aligned}$$

log sum	0.02297
log 16^4	4.81648
log x	5.20649
log 24	1.38021
$\log \frac{1}{24} \frac{d'^4 \alpha}{d t^4}$	3.82628

$$\frac{1}{24} \frac{d'^4 \alpha}{d t^4} = +0.00000067032$$

log sum	9.06819n
log 16^5	6.02060
log y	3.04759n
log 120	2.07918
$\log \frac{1}{120} \frac{d'^5 \alpha}{d t^5}$	0.96841n

$$\frac{1}{120} \frac{d'^5 \alpha}{d t^5} = -0.00000000062985$$

log sum	9.84386n
log 16^6	7.22472
log z	2.61914n
log 720	2.85733
$\log \frac{1}{720} \frac{d'^6 \alpha}{d t^6}$	9.76181n

$$\frac{1}{720} \frac{d'^6 \alpha}{d t^6} = -0.00000000005779$$

These values of the partial differential coefficients represent exactly all the values of $Y_0 - Y_4$ given on page 267 for every year from 1875 to 1803.

The residuals are as follows:

Epoch.	t	Assumed Values of $Y_0 - Y_4$	Computed Values of $Y_0 - Y_4$	Δ	Epoch.	t	Assumed Values of $Y_0 - Y_4$	Computed Values of $Y_0 - Y_4$	Δ
		s.	s.	s.			s.	s.	s.
1867	8	−0.002	−0.002	+0.000	1803	72	− 0.207	− 0.212	+0.005
1859	16	−0.001	−0.001	+0.000	1795	80	− 2.577	− 2.579	+0.002
1851	24	+0.024	+0.024	+0.000	1787	88	− 7.703	− 7.745	+0.042
1843	32	+0.102	+0.102	+0.000	1779	96	−17.276	− 17.279	+0.003
1835	40	+0.256	+0.256	+0.000	1771	104	−33.558	− 33.690	+0.132
1827	48	+0.480	+0.480	+0.000	1763	112	−59.531	− 60.043	+0.612
1819	56	+0.681	+0.681	+0.000	1755	120	−98.449	−100.391	+1.942
1811	64	+0.612	+0.612	+0.000					

We shall find in a similar manner from the sixteen-year intervals the following values of the coefficients of t, t^2, t^3, &c., from the residuals $Y_0 - Y_4$ from 1875 to 1955, given on page 268, viz.:

Epoch.	t	$Y_0 - Y_4$	Epoch.	t	$Y_0 - Y_4$
		s.			s.
1891	16	$a = +0.002$	1923	48	$c = -\ 1.785$
1907	32	$b = -0.206$	1939	64	$d = -\ 7.991$
			1955	80	$e = -25.308$

Whence by equations (64)

	Logarithms.
$\frac{d'\alpha}{dt} = +\overset{s.}{0}.0014106$	7.14941
$\frac{1}{2}\frac{d'^2\alpha}{dt^2} = -0.000033305$	5.92251n
$\frac{1}{6}\frac{d'^3\alpha}{dt^3} = +0.00000093590$	3.97122
$\frac{1}{24}\frac{d'^4\alpha}{dt^4} = +0.000000097911$	2.99083
$\frac{1}{120}\frac{d'^5\alpha}{dt^5} = -0.0000000089646$	1.95253

These values of the coefficients will reproduce all the values of $Y_0 - Y_4$ from 1875 to 1955, within the limits of the second decimal place.

As a further illustration of the degree of approximation to which the correspondence of the exact values Y_0 with the results given by equations (64) can be carried, we select the following values of $Y_0 - Y_{12}$ given on page 281, $J = \frac{d^4\alpha}{dt^4}$ and $w = 16$.

Epoch.	t	$Y_0 - Y_{12}$
		s.
1851	24	$a = -0.004$
1827	48	$b = +0.012$
1803	72	$c = +0.106$
1779	96	$d = +0.396$
1755	120	$e = +1.796$

Whence by equations (64)

$$\frac{d'\alpha}{dt} = +\overset{s.}{0}.0057304 \qquad \frac{1}{24}\frac{d'^4\alpha}{dt^4} = -\overset{s.}{0}.00000017733$$

$$\frac{1}{2}\frac{d'^2\alpha}{dt^2} = -0.0051764 \qquad \frac{1}{120}\frac{d'^5\alpha}{dt^5} = +0.00000000077027$$

$$\frac{1}{6}\frac{d'^3\alpha}{dt^3} = +0.000015143$$

These coefficients satisfy the values of $Y_0 - Y_{12}$ given on p. 281 as follows:

Epoch.	t	Assumed Values of $Y_0 - Y_{12}$	Computed Values of $Y_0 - Y_{12}$	Δ
		s.	s.	s.
1851	24	−0.004	−0.004	+0.000
1843	32	+0.000	+0.000	+0.000
1835	40	+0.004	+0.004	+0.000
1827	48	+0.012	+0.012	+0.000
1819	56	+0.028	+0.027	+0.001
1811	64	+0.060	+0.069	−0.009
1803	72	+0.106	+0.106	+0.000
1795	80	+0.170	+0.159	+0.011
1787	88	+0.263	+0.241	+0.022
1779	96	+0.396	+0.396	+0.000
1771	104	+0.604	+0.657	−0.053
1763	112	+0.990	+1.095	−0.105
1755	120	+1.796	+1.796	+0.000

It is evident that the interval of twenty-four years is in this case too great to allow an exact reproduction of the intermediate values of $Y_0 - Y_{12}$. We employ, therefore, equations (66) with the interval $w = 20$, in which it is necessary to compute the additional values for $t = 20$, $t = 60$, and $t = 100$.

DATA FOR THE COMPUTATION OF THE COEFFICIENTS.

Epoch.	t	$Y_0 - Y_{12}$
		s
1855	20	$a = -0.002$
1835	40	$b = +0.004$
1815	60	$c = +0.043$
1795	80	$d = +0.170$
1775	100	$e = +0.480$
1755	120	$f = +1.796$

Whence by equations (66)

$$\frac{d' \alpha}{d t} = -0\overset{s.}{.}005808 \qquad \frac{1}{24}\frac{d'^4 \alpha}{d t^4} = +0\overset{s.}{.}00000050287$$

$$\frac{1}{2}\frac{d'^2 \alpha}{d t^2} = +0.00064775 \qquad \frac{1}{120}\frac{d'^5 \alpha}{d t^5} = -0.0000000043880$$

$$\frac{1}{6}\frac{d'^3 \alpha}{d t^3} = -0.000026536 \qquad \frac{1}{720}\frac{d'^6 \alpha}{d t^6} = +0.000000000014714$$

The values of $Y_0 - Y_{12}$ are now represented as follows:

Epoch.	t	Assumed Values of $Y_0 - Y_{12}$	Computed Values of $Y_0 - Y_{12}$	Δ
		s.	s.	s.
1855	20	−0.002	−0.002	+0.000
1851	24	−0.004	−0.004	+0.000
1843	32	+0.000	+0.000	+0.000
1835	40	+0.004	+0.004	+0.000
1827	48	+0.012	+0.013	−0.001
1819	56	+0.028	+0.030	−0.002
1815	60	+0.043	+0.043	+0.000
1811	64	+0.060	+0.057	+0.003
1803	72	+0.106	+0.088	+0.018
1795	80	+0.170	+0.170	+0.000
1787	88	+0.263	+0.256	+0.007
1779	96	+0.396	+0.384	+0.012
1775	100	+0.480	+0.480	+0.000
1771	104	+0.604	+0.611	−0.007
1763	112	+0.990	+1.030	−0.040
1755	120	+1.796	+1.796	+0.000

In the choice of the interval w, it will be safe to assume any value not exceeding 15 when equations (64) and (66) are employed. When the interval does not exceed ten years, the correspondence will in any case be exact for the intermediate dates.

The following residuals, $Y_0 - Y_3$, will indicate how far beyond the last value of the series the correspondence with the exact values may continue. They were computed from equations (66), using slightly different elements from those thus far employed, in which $w = 10$. The correspondence for sixty years was found to be exact in every case. After this time, the assumed and the computed values were found to be as follows:

Epoch.	t	Assumed Values of $Y_0 - Y_3$	Computed Values of $Y_0 - Y_3$	Δ
		s.	s.	s.
1810	65	+ 61.27	+ 61.26	+0.01
1805	70	+ 82.02	+ 81.98	+0.04
1800	75	+107.38	+107.32	+0.06
1795	80	+137.88	+137.74	+0.14
1790	85	+174.06	+173.75	+0.31
1785	90	+216.44	+215.81	+0.63
1780	95	+265.51	+264.36	+1.15
1775	100	+321.75	+319.81	+1.94

It is obvious from this example, that there is no decided gain in the selection of residuals beyond the order $Y_0 - Y_3$ for the application of this method, unless the final values $Y_0 - Y_{12}$ are chosen. It will be seen that at the end of 15 years after the last date of the series the error amounts to only $0^s.06$. The computation will be expeditiously performed by making $w = 10$, and by changing the equinox at intervals of sixty years.

Collecting our results, we have, from pages 260, 261, and page 295, for any value of $+t$ between 1875 and 1955:

$$\begin{aligned}
\alpha = \alpha_0 &+ [+73\overset{s.}{.}514324 + 0\overset{s.}{.}0014106]\, t \\
&+ [-\ 0.15118944 - 0.000033305]\, t^2 \\
&+ [-\ 0.000339577 + 0.00000093590]\, t^3 \\
&+ [+\ 0.0000034726 - 0.0000000977911]\, t^4 \\
&\qquad\qquad + [+\ 0.0000000089646]\, t^5.
\end{aligned}$$

In like manner, we have from pages 260–264, and from page 296, for any value of $-t$ between 1875 and 1755, t being taken with the position sign in the computation of the secondary terms:

$$\begin{aligned}
\alpha = \alpha_0 &+ [+73\overset{s.}{.}514324 + 0\overset{s.}{.}005808]\, t \\
&+ [-\ 0.15118944 + 0.00064775]\, t^2 \\
&+ [-\ 0.000339577 + 0.000026536]\, t^3 \\
&+ [+\ 0.0000034726 + 0.00000050287]\, t^4 \\
&+ [-\ 0.000000005169 + 0.0000000043880]\, t^5 \\
&+ [-\ 0.00000000005554 + 0.000000000014714]\, t^6 \\
&\quad + 0.0000000000003173\, t^7 \\
&\quad + 0.0000000000000000163\, t^8 \\
&\quad - 0.00000000000000000845\, t^9 \\
&\quad + 0.000000000000000000259\, t^{10} \\
&\quad + 0.0000000000000000000000119\, t^{11} \\
&\quad - 0.00000000000000000000000080\, t^{12}
\end{aligned}$$

The second part of this paper will be comprised under the following subdivisions:—

(*a.*) Treatment of the proper motion for close polar stars.

(*b.*) Yearly ephemerides of all stars within 3° of the pole, between the limits 1860 and 1885.

(*c.*) Tabular values of the terms U^{I}, U^{II}, U^{III}, U^{IV}, U^{V}, &c., carried as far as will be necessary to give the exact reduction for 40 years.

(*d.*) Tabular values of the proper motions at intervals of 8 years for close polar stars, and at intervals of 20 years for all other stars.

(*e.*) Data for the reduction of the different catalogues employed to the system of Publication XIV.

(*f.*) Tabular values of the systematic relations between the catalogue of Publication XIV. and the different catalogues compared.

(*g.*) Final catalogues of 130 stars resulting from this discussion.

(*h.*) Comparison of the final catalogue with the various catalogues from which it has been derived.